

COMPUTATION OF FREE SOLIDARITY VALUE WITH ITS CHARACTERIZATION

Abstract

In this paper, we introduce a notion of free solidarity value with partial participation of solidary players that takes into account situations of solidary groups. And we construe a new value function for solidary and non-solidary players with its existence and uniqueness. In this regard, players can choose to mitigate the loss of their counterparts. We also add two new axioms, partial positivity and unreserved allocations of solidary players for the characterization of the value. We adhere to a unanimity game that forms a basis for games involving Solidarity and Shapley values.

Keywords: Cooperative game, Transferable Utility (TU), Solidarity Value, Shapley Value, Unanimity game.

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I. INTRODUCTION

A basic feature of an evolving society is that of people coming together as groups or unions. This leads to better productivity as well as an understanding to share the profit fairly. In particular, we consider the players are volunteering to form a solidarity group as highlighted in the working paper [5] titled "The free solidarity value" of Dhaou and Ziad. Wherein they considered that two types of players; solidary and non-solidary can co-exist without harming the interest of any group. The Shapley value [10] and the solidarity value [7] are employed for the payoffs of the non-solidary and solidary players respectively. Sprumont (1990) [11] introduced the Solidarity value. Nowak and Radzik characterized this solidarity value by replacing the null player axiom with the A-null player condition. The Shapley value is individualistic in nature. On the other hand, the Solidarity value expresses a certain degree of solidarity among the players. The first non-symmetric generalization of a value in coalitional form games with transferable utility is due to Shapley(1953a) [9]. Aumann and Dreze (1974) [1] and Owen (1977) [8] generalized the Shapley value for games with coalition structure. Hart and Kurz (1983) [4] developed a coalition structure value that in general takes certain situations where players can opt to join forces and in others act separately. There have been several papers employing the Shapley and Solidarity values with a coalition structure. Kamijo [6] introduced the two-step Shapley value and gave an axiomatization. This value was given another characterization by Calvo and Gutierrez [2] explicitly introduced the solidarity principle. In the Owen value, players interact at two levels, first among unions and then within unions. In both levels, the payoffs are given by the Shapley value. Calvo and Gutierrez [3] modelled the Shapley-solidarity value following the same approach as Owen. X.-F. Hu, D.-F. Li [13] proposed another axiomatization of the Shapley-Solidarity value [3] for coalitional structure TU- games. Su, Liang et.al [12] provided cooperative and non-cooperative interpretations of the Shapley-Solidarity value. The notion of free solidarity does not follow the classic concept of Owen [8] as well as the Shapley solidarity value [3], which follows a different approach though solely based on the Shapley and Solidarity distribution.

Dhaou and Ziad, principally, attempted to highlight solidarity that can exist in a social structure in situations where players are free to choose to form a solidary group. This voluntary formation of a group of solidary players sets the game in motion. But there also may be a situation where at least one or more members of the solidary group need to opt out of internal or external factors. In those situations, it is only fair to implement the Shapley value distribution otherwise some of the members of the solidary group need to take the extra burden for the missing ones. Dhaou and Ziad defined a basis for the game. However, we find that the value function constructed from the unanimity games does not correspond totally with the value function of the free solidarity value. Whereas for specific unanimity games, equality holds. Evidently, this restricts an efficient payoff for all the unanimity games. However, it is reasonable to assume this restriction is valid as the game structure depends on the solidary group of players, so efficiency fails to hold when all solidary players are not present. So, in light of this, we reconsider the payoffs of players when all solidary players are not present. In this paper, first, we analyze the unanimity games constructed as a basis game by Dhaou and Ziad [5]. We find that the value function based on the unanimity games agrees partially with the value function employing the Shapley and Solidarity values as defined in their paper. Albeit, for restricted unanimity game equality, holds it fails for the remaining games. We give a restriction that the unanimity games will strictly follow the Shapley distribution when all solidary players are not in a game that allows the solidary players to be treated as non-solidary players. And for which we

define the notion of partial participation of solidary players. This provides an effective measure to deal with efficiency thereby at the same time allowing free association without making it obligatory. It is relevant to ask how a solidarity group would be sustainable if the interests of some members are provided at the expense of other members.

So, our attempt in this paper is to provide fair distribution when some members of the solidary group fail to participate. We organize the paper as follows. Section 2 emphasizes on the mathematical preliminaries and all related values namely, the Shapley and the Solidarity values. Section 3 discusses on the free solidarity value as defined in their paper and further reconstructs the definition of the unanimity games. In section 4, we define partial participation of solidary players and also provide additional axioms called partial positivity and unreserved allocation of solidary players along with the existing axioms in the characterization of the free solidarity value.

However, we find that the value function constructed from the unanimity game does not correspond totally with the value function of the free solidarity value as defined in the paper. Whereas for specific unanimity games, equality holds. This restricts an efficient payoff for all the unanimity games. However, it is reasonable to assume this restriction is valid as the game structure depends on the solidary group of players, so efficiency fails to hold when all solidary players are not present. So, in light of this, we reconsider the payoffs of players when all solidary players are not present.

II. PRELIMINARIES

Given a finite set $N = \{1, 2, \dots, n\}$ of players. 2^N is the set of all subsets S of N . Any non-empty subset S of N is said to be a coalition. The set N is the grand coalition. A transferable utility (TU) game on a set N is a characteristic function $v: 2^N \rightarrow \mathbb{R}$ which assigns to each coalition S a real number denoted by $v(S)$, such that $v(\emptyset) = 0$. $v(S)$ is the worth of the coalition S . The cardinality of any non-empty coalition T, S, N etc can be denoted by t, s, n . A cooperative game with transferable utility (TU) is represented by (N, v) or simply v . The set of all games for a player set N is denoted by \mathcal{G}^N .

Consider any two games $v, u \in \mathcal{G}^N$, $\alpha \in \mathbb{R}$. Under the usual operations of addition and scalar multiplication, we can define, $(v + u)(S) = v(S) + u(S)$ and $(\alpha v)(S) = \alpha v(S)$ where $S \in 2^N \setminus \emptyset$. The set \mathcal{G}^N forms a vector space with dimension $2^N - 1$. A value is a mapping $\Phi: \mathcal{G}^N \rightarrow \mathbb{R}^n$ defined by $\Phi(v) = (\phi_1(v), \dots, \phi_i(v), \dots, \phi_n(v))$. For any game $v \in \mathcal{G}^N$, the mapping determines a unique payoff vector in \mathbb{R}^n . The vector represents the payoffs to each player $i \in N$ by taking into account their role in the game.

For any game $v \in \mathcal{G}^N$ and every $S \subseteq N$, the marginal contributions of a player i to a coalition S denoted by $m_i(v, S)$ is given by

$$m_i(v, S) = \begin{cases} v(S) - v(S \setminus i) & \text{if } i \in S \\ v(S \cup i) - v(S) & \text{if } i \notin S \end{cases}$$

In Nowak and Radzik [7], the average marginal contribution of a coalition S in a game denoted by $m^{av}(v, S)$ and is given by $m^{av}(v, S) = \frac{1}{|S|} \sum_{i \in S} (v(S) - v(S \setminus i)) = \frac{1}{|S|} \sum_{i \in S} m_i(v, S)$.

In Dhaou and Ziad [5], the average marginal contribution of a coalition S in a game is denoted by \tilde{m}^{av} and is given by.

$$\tilde{m}^{av}(v, S) = \frac{1}{|S \cap S^*|} \sum_{i \in S \cap S^*} (v(S) - v(S \setminus i)) = \frac{1}{|S \cap S^*|} \sum_{i \in S \cap S^*} m_i(v, S)$$

A game $v \in \mathcal{G}^N$ is said to be monotonic if $v(T) \leq v(S)$ for any $T, S \subset N$ such that $T \subseteq S$.

In a game $v \in \mathcal{G}^N$, two players are said to be symmetric if $v(S \cup i) = v(S \cup j) \forall S \subseteq N \setminus \{i, j\}$. If $v(S \cup i) = v(S)$, $\forall S \subseteq N \setminus i$ then $i \in N$ is a null player. If $\tilde{m}^{av}(v, S) = 0$ for some i which belongs to every coalition S and $S \subseteq N$ then i is A-null player.

Some of the properties of a value function Φ in \mathcal{G}^N are as follows:

1. Efficiency: For $v \in \mathcal{G}^N$, $\sum_{i \in N} \Phi_i(N, v) = v(N)$.
2. Additivity: For $v, u \in \mathcal{G}^N$, $\Phi_i(N, v + u) = \Phi_i(N, v) + \Phi_i(N, u)$.
3. Symmetry: If $i, j \in N$ are symmetric players in $v \in \mathcal{G}^N$ then $\Phi_i(N, v) = \Phi_j(N, v)$.
4. Null-player axiom: If $i \in N$ in $v \in \mathcal{G}^N$ is a null player then $\Phi_i(N, v) = 0$.
5. A-null player axioms: If $i \in N$ in $v \in \mathcal{G}^N$ is an A-null player then $\Phi_i(N, v) = 0$.

Players with zero contribution are highlighted in both the Shapley value [10] and the Solidarity value [7] or simply non-productivity is highlighted by the null player and A-null player conditions respectively. In the null player axiom, a player will get a zero payoff if all her marginal contributions in a game are zero. That is to label a player as strictly unproductive. In contrast, in the A-null player axiom a player will get a zero payoff when the average productivity of all the coalitions to which she belongs to is zero. This takes into consequence the presence of a player in a coalition before discarding her as unproductive.

1. The Shapley value: The Shapley value [10], $\Phi_{iS}(N, v)$ which gives the payoff for every player $i \in N$ is given by:

$$\Phi_{iS}(N, v) = \sum_{S \subseteq N, i \in S} \frac{(n-s)!(s-1)!}{n!} v_i(S) \quad \forall i \in N.$$

The Shapley value was characterized using the axioms of efficiency, additivity, symmetry and null-player axiom. The Shapley value can also be defined by using the unanimity games. A unanimity game u_T with $\emptyset \neq T \subseteq N$ is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

The Shapley value in terms of each vector of the unanimity game u_T is given by:

$$Sh(N, u_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

The unanimity games $(u_T)_{\emptyset \neq T \subseteq N}$ form a basis for \mathcal{G}^N . It has been established that any game can be written as a linear combination of the unanimity games and coefficients of which is Δ_T . That is, $v = \sum_{\emptyset \neq T \subseteq N} \Delta_T u_T$ where $\Delta_T = \sum_{T \subseteq S} (-1)^{s-t} v(T)$ is the *Harsanyi dividend*.

Thus, the Shapley value in the dividend form is defined as $S\psi(N, v) = \Delta_T \frac{1}{|T|}$ for every $i \in N$.

Shapley [10] has also given the following theorem

Theorem 1: A value Ψ on \mathcal{G}^N satisfies efficiency, additivity, symmetry, and the null player axiom if, and only if, Ψ is the Shapley value.

2. **The Solidarity value:** Sprumont [11], defined the solidarity value Φ_{isd} in a recursive manner as follows

$$Sd_i(N, v) = \frac{1}{s} m^{av}(v, S) + \sum_{j \in S \setminus i} \frac{1}{s} Sd_i(S \setminus j, v). \quad i \in S \subseteq N, \quad \text{Firstly, } Sd_i(\{i\}, v) = v(i) \quad \forall i \in N.$$

Nowak and Radzik [7] gave another version of the solidarity value and characterized the value with the axioms of efficiency, additivity, symmetry, and A-null player condition. The difference in the characterization with the Shapley value is in the axiom of the A-null player which replaces the null player axiom.

The Solidarity value in Nowak and Radzik [7] is defined as

$$Sd_i(N, v) = \sum_{i \in S} \frac{(n-s)!(n-1)!}{n!} m^{av}(v, S) \quad \forall i \in N.$$

In the authors also constructed a new basis game for \mathcal{G}^N . The game is defined as

$$b_T = \begin{cases} \binom{|S|}{|T|}^{-1} & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

For the game $(N, b_T)_{\emptyset \neq T \subseteq N}$ they showed that all the players in $N \setminus T$ are A-null players. The solidarity value based on this basis game is as follows:

$$Sd_i(N, b_T) = \begin{cases} \frac{1}{|T|} b_T(N) & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

Nowak and Radzik [7] have given the following theorem

Theorem 2: A value Ψ on \mathcal{G}^N satisfies efficiency, additivity, symmetry, and the A-null player axiom if and only if Ψ is the solidarity value.

- 3. The Free Solidarity value:** Dhaou and Ziad (working paper, [8]) defined the free solidarity value wherein they considered two types of players, solidary and non-solidary players. The group of solidary players is fixed from the onset of the game. The group of solidary players is denoted by S^* .

They defined the unanimity game for the Free Solidarity value:

$$B_{(T,S^*)}(S) = \begin{cases} \frac{|T \cap S^*|}{|T|} \left(\frac{|S \cap S^*|}{|T \cap S^*|} \right)^{-1} + \frac{|T \setminus (T \cap S^*)|}{|T|} & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The authors showed that the unanimity games $(B_{(T,S^*)})_{\emptyset \neq T \subseteq N}$ exhibit the following properties:

If $T = S$ then $B_{(T,S^*)}(S) = 1$, (ii) If $T \subset S$ and $S^* = \emptyset$, then $B_{(T,\emptyset)}(S) = 1$, (iii) $B_{(T,S^*)}$ is a basis for \mathcal{G}^N .

The free solidarity value based on the unanimity games is defined as follows:

$$\Psi_i(N, B_{(T,S^*)}, S^*) = \begin{cases} \frac{1}{|T|} \left(\frac{|S^*|}{|T \cap S^*|} \right)^{-1} & \text{if } i \in T \cap S^* \\ \frac{1}{|T|} & \text{if } i \in T \setminus (T \cap S^*) \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Dhaou B., Ziad, A. [5] have given the following theorem for free solidarity value.

Theorem 3: A value Ψ on \mathcal{G}^N satisfies efficiency, additivity, conditional symmetry, conditional null player condition, and unaffected allocation of non-solidary players axioms if and only if Ψ is the free solidarity value.

In the free Solidarity value [5], the authors assert that the existence of an exogenous coalition of solidary players promotes fairness in the payoffs of the players. The payoffs are considered according to the players being solidary or non-solidary. So, the authors distinguish the non-solidary and solidary players by accordingly considering their payoffs to be the Shapley value and the Solidarity value respectively. From this, it follows that the contributions of the players in a game are regarded as marginal contribution or average marginal contribution depending on the player being non-solidary or solidary.

Thus, if $\forall i \in S \subseteq N \setminus \emptyset$, $S \cap S^* \neq \emptyset$, then, $m_i(v, S) = v_{\mathcal{S}} - v_{\mathcal{S} \setminus i}$. And, if $i \in S \setminus S \cap S^*$

$$\text{or } \tilde{m}^{av}(v, S) = \frac{1}{|S \cap S^*|} \sum_{i \in S \cap S^*} (v_{\mathcal{S}} - v_{\mathcal{S} \setminus i}) = \frac{1}{|S \cap S^*|} \sum_{i \in S \cap S^*} m_i(v, S), \text{ if } i \in S \cap S^*.$$

These contributions distinguish a null player and an A-null player. Null Players will get zero payoffs if their marginal contributions are zero and A-null players will get nothing if average marginal contributions are zero for every coalition the player belongs. The free solidarity value for an exogenous coalition S^* is defined as

$$\Phi_{ifs}(N, v, S^*) := \begin{cases} \sum_{i \in S^*} \frac{(n-s)!(s-1)!}{n!} [v(\xi) - v(S \setminus \{i\})] & i \notin S^* \\ \sum_{i \in S^*} \frac{(n-s)!(s-1)!}{n!} \frac{1}{|S \cap S^*|} [\sum_{k \in S \cap S^*} (v(\xi \setminus k)) - v(\xi \setminus k)] & \text{if } i \in S^* \end{cases} \quad (3)$$

$$\Phi_{ifs}(N, v, S^*) := \begin{cases} Sh_{ifs}(N, v) & \text{if } i \notin S^* \\ Sd_{ifs} & \text{if } i \in S^* \end{cases} \quad (4)$$

The following relations hold for equations (3) and (4)

- If $S^* = \emptyset$ then $\Phi^{fs}(N, v, S^*) = Sh^s(N, v)$
- If $S^* = \{i\}$ then $\Phi^{fs}(N, v, S^*) = Sh^s(N, v)$
- If $S^* = N$ then $\Phi^{fs}(N, v, S^*) = Sd^{fs}(N, v)$

We illustrate with an example that the definition Dhaou B., Ziad, A [8] in Eq.(3) does not in totally agree with the definition given in Eq.(2) given concerning the unanimity games.

Example 1: The matrix of the unanimity game $(N, B_{(T, S^*)}, S^*)$ based on the example, $N = \{1, 2, 3\}$; $S^* = \{2, 3\}$ is as follows:

$B_{(T, S^*)}(S)$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$B_{\{1\}}$	1	0	0	1	1	0	0
$B_{\{2\}}$	0	1	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$
{3}	0	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$
{1,2}	0	0	0	1	0	0	$\frac{3}{4}$
{1,3}	0	0	0	0	1	0	$\frac{3}{4}$
{2,3}	0	0	0	0	0	1	1
{1,2,3}	0	0	0	0	0	0	1

The payoff matrix for free solidarity value (Eq.2) in example 1 is as follows:

$B_{(T, S^*)}$	$i = 1$	$i = 2$	$i = 3$	$\Psi(N, B_{(T, S^*)}, S^*)$	$\sum_{i \in N} \Psi_i$
{1}	1	0	0	(1,0,0)	1
{2}	0	$\frac{1}{2}$	0	$(0, \frac{1}{2}, 0)$	$\frac{1}{2}$
{3}	0	0	$\frac{1}{2}$	$(0, 0, \frac{1}{2})$	$\frac{1}{2}$
{1,2}	$\frac{1}{2}$	$\frac{1}{4}$	0	$(\frac{1}{2}, \frac{1}{4}, 0)$	$\frac{3}{4}$

{1,3}	$\frac{1}{2}$	0	$\frac{1}{4}$	$(\frac{1}{2}, 0, \frac{1}{4})$	$\frac{3}{4}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$	$(0, \frac{1}{2}, \frac{1}{2})$	1
{1,2,3}	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1

Again, we apply the unanimity games on the definition of the free solidarity value (Eq.3) as given in the paper.

$B_{(T,S^*)}$	$i = 1$	$i = 2$	$i = 3$	$\Phi^{fs}(N, B_{(T,S^*)}, S^*)$	$\sum_{i \in N} \Phi_{ifs}$
{1}	1	0	0	(1,0,0)	1
{2}	0	$\frac{1}{2}$	0	$(0, \frac{1}{2}, 0)$	$\frac{1}{2}$
{3}	0	0	$\frac{1}{2}$	$(0, 0, \frac{1}{2})$	$\frac{1}{2}$
{1,2}	$\frac{5}{12}$	$\frac{1}{4}$	0	$(\frac{5}{12}, \frac{1}{4}, 0)$	$\frac{2}{3}$
{1,3}	$\frac{5}{12}$	0	$\frac{1}{4}$	$(\frac{5}{12}, 0, \frac{1}{4})$	$\frac{2}{3}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$	$(0, \frac{1}{2}, \frac{1}{2})$	1
{1,2,3}	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1

In the above payoff matrix, we see that efficiency does not hold for all the unanimity games. In general, the unanimity games are in partial agreement, restricted to those unanimity games which has all solidary players with one or more or all non-solidary players, including only all the solidary players. To maintain efficiency for all the unanimity games we need to consider the Shapley value distribution payoffs for the games that do not have all the solidary players. Which we modify the unanimity games by adding an element implementing the Shapley value.

In light of this, we implement the following Shapley distribution when players from the already formed solidary group move out of the solidary group.

$$\Phi_{ifs}(N, v, S^*) := \left\{ \sum_{i \in S^*} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S \setminus \{i\})] \right\} \quad \text{for } i \in N \quad (5)$$

III. DEFINITIONS

- Solidary group:** In a game, when players voluntarily agree to form a group to promote the common interest with mutual support of its members, we say the players form a solidary group. In this case, players are known as solidary players. We denote this coalition group by S^* .

2. **Non-Solidary group:** In a game, when players do not join the solidary group, they naturally give rise to a non-solidary group. In this case, players are known as non-solidary players.
3. **Partial participation of solidary players:** We introduce the notion of partial participation of solidary players in the context of unanimity games where all members of the solidary group do not participate. Then in all those unanimity games the remaining solidary players are treated as non-solidary players. In other words, after the voluntary formation of the solidary group, if certain players decide to step out of the solidary group, then for the particular game the remaining solidary group members will also be treated as non-solidary players.

Nonetheless, this does not change the spirit of free solidarity which forms the original notion of construction of the unanimity games in their paper. To maintain the solidarity property of the solidary group all members of the group should fully participate in the game as a solidary player. So, keeping the original axioms *we add two new axioms 5 and 7 that will compensate for the discrepancy that arises in the original construction of the unanimity games.*

4. **Axiomatic Characterization of the Free Solidarity Value:** Let $\Psi^{fs}(N, v, S^*)$ be a value of a cooperative TU-game \mathcal{G}^N . The following are the axioms for Ψ^{fs} :
 - **Efficiency (ϵ):** For any $(N, v, S^*) \in \mathcal{G}^N$, we have, $\sum_{i \in N} \Psi_{ifs}(N, v, S^*) = v(N)$.
 - **Additivity (\mathcal{A}):** For any $(N, v, S^*), (N', w, T^*) \in \mathcal{G}^N$ with $N = N', S^* = T^*$, we have, $\Psi_{ifs}(N, v + w, S^*) = \Psi_{ifs}(N, v, S^*) + \Psi_{ifs}(N, w, S^*)$.
 - **Conditional symmetry ($\mathcal{C} - \mathcal{S}$):** If $i, j \in N$ are symmetric players in (N, v, S^*) , then $\Psi_{ifs}(N, v, S^*) = \Psi_{jfs}(N, v, S^*)$ if either $i, j \in S^*$ or $i, j \in S \setminus S^*$.
 - **Partial Null Player Property ($\mathcal{P} - \mathcal{N}$):** For any game $(N, v, S^*) \in \mathcal{G}^N$, if $i \in S \setminus S^* \cap S^*$ has the null player property then $\Psi_{ifs}(N, v, S^*) = 0$. Moreover, if there is partial participation of solidary players then any $i \in N$ with the null player property has $\Psi_{ifs}(N, v, S^*) = 0$.
 - **Partial positivity ($\mathcal{P} - \mathcal{P}$):** For any game $(N, v, S^*) \in \mathcal{G}^N$, if there is no partial participation of solidary players then $\Psi_{ifs}(N, v, S^*) > 0$ for every $i \in S^*$.
 - **Unaffected allocation of non-solidary players:** When players decide freely to be solidary to form a solidary group S^* , it will neither affect the allocation of the players who have chosen to stay out of the group nor the value v . For every $S \subseteq N, v \notin S, S^*) = v(S, \emptyset)$ and $\forall i \notin S^* \neq \emptyset, \Psi_{ifs}(N, v, S^*) = \Psi_{ifs}(N, v, \emptyset)$.
 - **Unreserved allocation of solidary players:** When partial participation of solidary players in a game takes place due to some external or internal factors, all the solidary players are treated as non-solidary and hence their payoffs are given by the Shapley distribution.

$\Psi_{ifs}(N, v, S^* \setminus j) = Sh(N, v)$ where j denotes the number of non-participating solidary players.

Theorem 4: A value $\Psi^{fs}: \mathcal{G}^N \rightarrow R^n$ satisfies efficiency, additivity, conditional symmetry, partial null player property, partial positivity, unaffected allocation of non-solidary players, and unreserved allocation of solidary players axioms if and only if $\Psi^{fs} = \Phi^{fs}$, that is, Ψ^{fs} is the free solidarity value. We define the reconstructed unanimity game as follows:

Definition 4

$$B_{(T,S^*)}(S) = \begin{cases} \frac{|T \cap S^*|}{|T|} \left(\frac{|S \cap S^*|}{|T \cap S^*|} \right)^{-1} + \frac{|T \setminus (T \cap S^*)|}{|T|} & \text{if } T \subseteq S \text{ wkn all solidary players} \\ & \text{are in } T \\ 1 & \text{if } T \subseteq S \text{ wkn all solidary palyers are not in } T \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Lemma 1: The unanimity game $B_{(T,S^*)}(S)$, $\forall T \subseteq S$ with $T \neq \emptyset$, has the following properties:

- If $T = S$, then $B_{(T,S^*)}(S) = 1$.
- If $T \subset S$ with $S^* = \emptyset$, then $B_{(T,\emptyset)}(S) = 1$.
- If $T \subset S$ with $T \subset S^*$, then $B_{(T,S^*)}(S) = 1$.
- The unanimity games with $B_{(T,S^*)}$ with T containing all the solidary players have strictly $\tilde{m}^{av}(B_{(T,S^*)}, S) \neq 0$.
- In the unanimity games, if $i \in S^*$ and $i \in N \setminus T$ with T not containing all the solidary players $T \subseteq S$, then we have, $\tilde{m}^{av}(B_{(T,S^*)}, S) \neq 0$ for every solidary player $i \in T$, that is, $B_{(T,S^*)}(S) \neq \frac{1}{|S \cap S^*|} \sum_{j \in (S \cap S^*)} (B_{(T,S^*)}(S) - B_{(T,S^*)}(S \setminus j))$.

This implies that $m_i(B_{(T,S^*)}, S) = 0$ will hold for any $i \in N \setminus T$ having the null player property.

Proof: The proof follows immediately from the definition of the unanimity games. The parts 1, 2, 3 are direct. Also, the games $B_{(T,S^*)}$ with T containing all the solidary players have strictly $\tilde{m}^{av}(B_{(T,S^*)}, S) \neq 0$ for every solidary player $i \in T$. Hence, in this case, none of the solidary players will have the A-null player property. Again, by the definition of games $B_{(T,S^*)}$ with T containing all the solidary players will exhibit the null-player property if $i \in N \setminus T$ that is $m_i(B_{(T,S^*)}, S) = 0$.

Lemma 2: Let (N, v, S^*) be a game and S^* be the coalition group of solidary players with $|S^*| \geq 2$. If there is no partial participation of solidary players in a game, then no player in S^* is an A-null player.

Proof: Clearly, it follows from part 4 of the lemma 3.3.

Remark 1: We conclude that the A-null player condition will not hold for the solidary players for which we imply that a player will only have the null player property. So, all players regardless of being solidary or non-solidary will only possess the null player property.

Lemma 3: The set $\mathcal{B} = \{B_{(T,S^*)}(S) | T \neq \emptyset\}$ of games is a basis for the vector space \mathcal{G}^N .

We follow the proof given in Nowak and Radzik [7]. Firstly, \mathcal{G}^N is a K -dimensional vector space. $K = 2^n - 1$ is the number of possible coalitions of the player set N excluding the empty set \emptyset . Suppose that T_1, T_2, \dots, T_k be a fixed sequence of non-empty coalitions of the player set N satisfying $1 = |T_1| \leq |T_2| \leq \dots \leq |T_k| = n$. We define a $K \times K$ Matrix $A = [a_{ij}]$ as $a_{ij} = B_{(T,S^*)}(S_j)$, for $j, j = 1, 2, \dots, k$. By the definition of the unanimity game (Equation 6), it follows that matrix A is a triangular matrix with all diagonal entries equal to 1. This implies $|A| \neq 0$. Hence, the set $\{B_{(T,S^*)}; i = 1, 2, \dots, k\}$ forms k linearly independent vectors in the vector space \mathcal{G}^N . So, the set \mathcal{B} is a basis.

Lemma 4: If the value Ψ^{fs} satisfies efficiency, additivity, symmetry, partial null player property, partial positivity, unaffected allocation of non-solidary players, and unreserved allocation of solidary players axioms, then for every player $i \in N$, non-empty coalition T , and a real constant β we have the following two cases:

Case 1: When all solidary players are in T

$$\Psi_{ifs}(N, \beta B_{(T,S^*)}, S^*) = \begin{cases} \left(\frac{\beta}{|T|} \left(\frac{|S^*|}{|T \cap S^*|} \right)^{-1} \right) & \text{if } i \in T \cap S^* \\ \frac{\beta}{|T|} & \text{if } i \in T \setminus (T \cap S^*) \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Case 2: When all solidarity players are not in T

$$\Psi_{ifs}(N, \beta B_{(T,S^*)}, S^*) = \begin{cases} \frac{\beta}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

Proof: Suppose that $\emptyset \neq T \subseteq N$. As β is any real number, it follows that $\beta B_{(T,S^*)}$ is a game. Consider $\beta = 0$, then the lemma follows from efficiency and symmetry axioms. Assume that $\beta \neq 0$. For $T \subseteq N$, in both the cases of T consisting of all solidary and not all solidary players, every player $i \in N \setminus T$ will be a null player in the game $\beta B_{(T,S^*)}$ by lemma 3.4. Hence $\Psi_{ifs}(N, \beta B_{(T,S^*)}, S^*) = 0$ for each $i \in N \setminus T$. By efficiency, the two cases can be derived.

IV. ILLUSTRATION USING EXAMPLE 1

The matrix of the new unanimity game $(N, B_{(T,S^*)}, S^*)$ based on example 1, $N = \{1, 2, 3\}$; $S^* = \{2, 3\}$ is as follows:

$B_{(T,S^*)}(S)$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{1}	1	0	0	1	1	0	1
{2}	0	1	0	1	0	1	1
{3}	0	0	1	0	1	1	1
{1,2}	0	0	0	1	0	0	1
{1,3}	0	0	0	0	1	0	1

{2,3}	0	0	0	0	0	1	1
{1,2,3}	0	0	0	0	0	0	1

Then, the payoff matrix for free solidarity value (Eq. 7 and 8) in example 1 is as follows:

$B_{(T,S^*)}$	$i = 1$	$i = 2$	$i = 3$	$\Psi^{fs}(N, B_{(T,S^*)}, S^*)$	$\sum_{i \in N} \Psi_{ifs}$
{1}	1	0	0	(1,0,0)	1
{2}	0	1	0	(0,1,0)	1
{3}	0	0	1	(0,0,1)	1
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0	$(\frac{1}{2}, \frac{1}{2}, 0)$	1
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$	$(\frac{1}{2}, 0, \frac{1}{2})$	1
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$	$(0, \frac{1}{2}, \frac{1}{2})$	1
{1,2,3}	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1

Again, we apply the unanimity games to the definition of the free solidarity value (Eq. 3) as given in the paper.

$B_{(T,S^*)}$	$i = 1$	$i = 2$	$i = 3$	$\Phi^{fs}(N, B_{(T,S^*)}, S^*)$	$\sum_{i \in N} \Phi_{ifs}$
{1}	1	0	0	(1,0,0)	1
{2}	0	1	0	(0,1,0)	1
{3}	0	0	1	(0,0,1)	1
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0	$(\frac{1}{2}, \frac{1}{2}, 0)$	1
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$	$(\frac{1}{2}, 0, \frac{1}{2})$	1
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$	$(0, \frac{1}{2}, \frac{1}{2})$	1
{1,2,3}	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1

Remark: By the axiom of additivity, lemma 4.3 and lemma 4.4, we have the following lemma.

Lemma 5 : Any value that satisfies all axioms 4.1 is a linear mapping from \mathcal{G}^N into R^n

V. PROOF OF THEOREM 4 (EXISTENCE AND UNIQUENESS OF THE VALUE FUNCTIONS)

- 1. Existence:** We first prove the existence of a value that satisfies the axioms 4.1. Firstly, the efficiency of Φ^{fs} is justified by the fact that Φ^{fs} is efficient for any unanimity game $B_{(T,S^*)}$. Let $v \in \mathcal{G}^N$. Then, there exist constants $\lambda_T, \emptyset \neq T \subset N$, such that $v = \sum_{\emptyset \neq T \subset N} \lambda_T B_{(T,S^*)}$.

To show that Φ^{fs} is efficient, we use the linearity of Φ^{fs} .

$$\sum_{i \in N} \Phi_{ifs}(v) = \sum_{\emptyset \neq T \subset N} \lambda_T \sum_{i \in N} \Phi_{ifs}(B_{(T,S^*)}) = \sum_{\emptyset \neq T \subset N} \lambda_T B_{(T,S^*)}(N) = v(N).$$

Thereby, showing that Φ^{fs} is an efficient value. It also follows that the value function Φ^{fs} satisfies conditional symmetry, partial null player condition, partial positivity, unaffected allocation of non-solidary players, and unreserved allocation of solidary players. Φ^{fs} is additive as it is a linear mapping.

- 2. Uniqueness:** Let $\Psi^{fs}(N, B_{(T,S^*)})$ be a value function on \mathcal{G}^N satisfying the axioms 1 to 7. As β is any real number, it follows that $\beta B_{(T,S^*)}$ is a game.

If $i \notin T$ and $T \not\subseteq S$ then $\Psi_{ifs}(N, B_{(T,S^*)}) = 0$. If $i \notin T$ and $T \subseteq S$, that is, for any $i \notin N \setminus T$, $\beta B_{(T,S^*)}(S) = \beta B_{(T,S^*)}(S \cup i)$ which implies that $m_i(\beta B_{(T,S^*)}, S) = 0$.

Let $T_1 = T \setminus T \cap S^*$ and $T_2 = (T \cap S^*)$. For any two players $i, j \in T \setminus (T \cap S^*)$ or $i, j \in T \cap S^*$ with $i, j \notin S$, we have $T_1, T_2 \not\subseteq S$ which implies $T_1, T_2 \not\subseteq S \cup \{i\}$ and $T_1, T_2 \not\subseteq S \cup \{j\}$. It follows that for $T = T_1 \cup T_2$, $\beta B_{(T,S^*)}(S \cup i) = \beta B_{(T,S^*)}(S \cup j)$. Then, by conditional symmetry $\Psi_{ifs}(N, \beta B_{(T,S^*)}) = \Psi_{jfs}(N, \beta B_{(T,S^*)})$ for every $i, j \in T$.

Moreover $\sum_{i \in T} \Psi_{ifs}(N, \beta B_{(T,S^*)}) = |T| \Psi_{ifs}(N, \beta B_{(T,S^*)})$ for any $i \in T$.

If there is no partial participation of solidary players then $\tilde{m}^{av}(S_{(T,S^*)}, S) \neq 0$ for every solidary player $i \in T$. This implies by the partial positivity axiom that $\Psi_{ifs}(N, \beta B_{(T,S^*)}) > 0$.

As the unaffected allocation holds for non-solidary players, in a similar way the unreserved allocation of solidary players holds. By definition of the unanimity game, if in a game solidary players withdraw from the formed solidary group, they are to be treated as non-solidary. Hence no reservation is set for them as well. Hence by the axiom, partial participation of the solidary players will make it necessary for their payoffs to follow the Shapley distribution which is indicated to be $\Psi_{ifs}(N, \beta B_{(T,S^*)}, S^* \setminus j) = Sh(N, \beta B_{(T,S^*)})$ where j denotes the number of non-participating solidary players. More precisely, in terms of unanimity game $\beta B_{(T,S^*)}$ with T containing not all the solidary players.

By lemma 4.5, Ψ^{fs} is a linear mapping. Applying lemma 4.4 to both Ψ^{fs} and Φ^{fs} , we observe that $\Psi^{fs}(B_{(T,S^*)}) = \Phi^{fs}(B_{(T,S^*)})$ for each unanimity game $B_{(T,S^*)}$. Hence, $\Psi^{fs}(v) = \Phi^{fs}(v)$ for every game $v \in \mathcal{G}^N$.

VI. CONCLUSION

This paper concludes that solidarity value has a human element in the distribution of the payoffs among the players. The value ameliorates the condition of players who do not contribute anything substantial to the game. So, in the free solidarity value players opting to be solidary give free will support for players lacking direct contributions to the game. As for the non-solidary players, the Shapley value attributes players opting out of the solidarity act. This entitles them to a payoff that is based on their marginal contribution to coalitions which may form.

Further, we investigate the case of players who initially have followed the principle of solidarity and are unable to continue their pursuance of the act due to internal or external factors. As a consequence of this, each player involved in such coalitions is meted out payoffs based on the Shapley value. This maintains fairness and balance for each of the players in the coalition they form.

Thus, in this paper, we have taken into account the possible existence of solidary and non-solidary players without hindering the payoffs of any of the players. We have justified our results by computation presented in tables based on both the Shapley value and the Solidarity value. The future scope of this research is to apply this Solidary group and Non-Solidary group in supply chain management to reduce the total cost. This cost reduction will make it possible to form a solidary group in horizontal cooperation and a non-solidary group within vertical cooperation.

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