# A CONTEMPORARY CLASS OF DOUBLE INTEGRAL INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

**Abstract**

The study of double integrals plays a vital role in mathematics, particularly in the field of calculus. In recent years, there has been growing interest in exploring new techniques and approaches to evaluate double integrals using generalized hypergeometric functions. The aim of this paper is to provide the contemporary class of double integral. To achieve this, we employed the generalized hypergeometric functions established by Eslahchi and Masjed-Jamei (2015) with a very interesting integral due to Edward (1954). The results established in the paper are useful, easy to evaluate, and provide a versatile framework for solving complex mathematical problems.

**Keywords:** Generalized Hypergeometric Functions, Classical Summation Theorems, Double Integral.

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## I. INTRODUCTION

The contemporary class refers to a specific set of double integrals that can be evaluated by utilizing generalized hypergeometric functions. Prior to getting into the particulars of the contemporary class of double integrals, it is necessary to comprehend the concept of generalized hypergeometric functions. These are generalized versions of common hypergeometric series. Due to their capacity to represent solutions to linear differential equations with polynomial coefficients, they have a wide range of applications.

The following is a typical generalized hypergeometric series with $p$ numerator and $q$ denominator parameters is as follows:

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_p)_n}{(b_1)_n (b_2)_n ... (b_q)_n} \frac{z^n}{n!} \tag{1.1} $$

where no denominator parameter is supposed to be zero or negative integer, if any numerator parameter is zero or negative integer the series terminates.

The series in (1) converges for all values of z when $p < q$. When $p = q + 1$, the series is convergent if $|z| < 1$ and divergent if $|z| > 1$. Further if $p = q + 1$, it converges absolutely for $|z| = 1$ provided $\Re\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right] > 0$

Moreover $(a)_n$ is known as shifted factorial or Pochhammer's symbol defined as:

$$ (a)_n = \begin{cases} 1, & n = 0 \\ (a)(a+1)(a+2).......(a+n-1), & n \in N \end{cases} \tag{1.2} $$

Additionally, the results are very applicable and advantageous from an applications standpoint when a generalized hypergeometric function, ${}_{q+1}F_q$, is turned into a gamma function. The following classical summation theorems will be covered; therefore, the work should stand on its own [3,12,11]:

**1.** Gauss's summation theorem: For $\Re(c - a - b) > 0$

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.3} $$

**2.** Kummer's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix} ; -1\right] = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma(1+a)} \tag{1.4} $$

**3.** Gauss's second summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,b \\ \frac{1}{2}(1+a+b) \end{matrix} ; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)} \tag{1.5} $$

**4.** Bailey's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,1-a \\ b \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}(b+1)\right)}{\Gamma\left(\frac{1}{2}(a+b)\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)} \tag{1.6} $$

**5.** Dixon's summation theorem: For $\Re(a-2b-2c)>-2$

$$ {}_3F_2\left[\begin{matrix} a,b,c \\ 1+a-b,1+a-c \end{matrix} ; 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma\left(1-b-c+\frac{1}{2}a\right)}{\Gamma(1+a)\,\Gamma\left(1-b+\frac{1}{2}a\right)\Gamma\left(1-c+\frac{1}{2}a\right)\Gamma(1+a-b-c)} \tag{1.7} $$

**6.** Watson's summation theorem: For $\Re(2c-a-b)>-1$

$$ {}_3F_2\left[\begin{matrix} a,b,c \\ \frac{1}{2}(1+a+b),2c \end{matrix} ; 1\right] = \frac{\sqrt{\pi}\;\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)} \tag{1.8} $$

**7.** Whipple's summation theorem:

$$ {}_3F_2\left[\begin{matrix} a,1-a,b \\ c,2b-c+1 \end{matrix} ; 1\right] = \frac{\pi 2^{1-2b}\Gamma(c)\Gamma(2b-c+1)}{\Gamma\left(\frac{1}{2}(a+c)\right)\Gamma\left(b+\frac{1}{2}(a-c+1)\right)\Gamma\left(\frac{1}{2}(1-a+c)\right)\Gamma\left(b+1-\frac{1}{2}(a+c)\right)} \tag{1.9} $$

## II. RESULTS REQUIRED

In 2015 Masjed-Jamei and Eslahchi [5] generalized the above classical summation theorems by employing the following identity:

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} ; \mathrm{z}\right] = \sum_{k=0}^{m-1} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{\mathrm{z}^{\mathrm{k}}}{\mathrm{k}!} \times {}_{mp+1}F_{mq+m}\left[\begin{matrix} \vec{A}_{1,k}, \vec{A}_{2,k}, \ldots, \vec{A}_{p,k}, 1 \\ \vec{B}_{1,k}, \vec{B}_{2,k}, \ldots, \vec{B}_{q,k}, \vec{I}_{1,k} \end{matrix} ; \mathrm{m}^{(p-q-1)m} z^m\right] \tag{2.1} $$

Where,

$$ \vec{A}_{j,k} = \left( \frac{a_j + k}{m}, \frac{a_j + 1 + k}{m}, \ldots, \frac{a_j + m - 1 + k}{m} \right) $$

for $j = 1, 2, \ldots, p$

$$ \vec{B}_{j,k} = \left( \frac{b_j + k}{m}, \frac{b_j + 1 + k}{m}, \ldots, \frac{b_j + m - 1 + k}{m} \right) $$

for $j = 1, 2, \ldots, q$

$$ \vec{I}_{1,k} = \left( \frac{1+k}{m}, \frac{2+k}{m}, \ldots, \frac{m+k}{m} \right) $$

By employing above general case (2.1), generalization of classical summation theorems (1.3)- (1.9) are following:

**1.** The general case of the Gauss's summation theorem for any natural number m is as:

$$ \sum_{k=0}^{m-1} \frac{(a)_k (b)_k}{(c)_k k!} {}_{2m+1}F_{2m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, 1 \\ \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix} ; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.2} $$

provided $\Re(c-a-b) > 0$

**2.** The general case of Kummer's summation theorem for any natural number m is as:

$$ \sum_{k=0}^{m-1} \frac{(a)_k (b)_k (-1)^k}{(1+a-b)_k k!} {}_{2m+1}F_{2m}\left[\begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{b+k}{m}, \ldots, \frac{b+m-1+k}{m}, 1 \\ \frac{1+a-b+k}{m}, \ldots, \frac{a-b+m+k}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix} ; (-1)^m\right] = \frac{\Gamma\left(1+a-b\right)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1+a\right)\Gamma\left(1+\frac{a}{2}-b\right)} \tag{2.3} $$

**3.** The general case of Gauss's second summation theorem for any natural number m is as:

$$\sum_{k=0}^{m-1}\frac{(a)_k(b)_k 2^{-k}}{\left(\frac{1}{2}(a+b+1)\right)_k k!} \times {}_{2m+1}F_{2m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},1\\ \frac{k+(a+b+1)/2}{m},\ldots,\frac{m-1+k+(a+b+1)/2}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};2^{-m}\right] = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)} \tag{2.4}$$

**4.** The general case of Bailey's summation theorem for any natural number m is as:

$$\sum_{k=0}^{m-1}\frac{(a)_k(1-a)_k(2)^{-k}}{(c)_k k!}\,{}_{2m+1}F_{2m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{1-a+k}{m},\ldots,\frac{-a+m+k}{m},1\\ \frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};2^{-m}\right] = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{1}{2}(c-a+1)\right)} \tag{2.5}$$

**5.** The general case of Watson's summation theorem for any natural number m is as:

$$\sum_{k=0}^{m-1}\frac{(a)_k(b)_k(c)_k}{\left(\frac{1}{2}(a+b+1)\right)_k(2c)_k k!} \times {}_{3m+1}F_{3m}\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},\frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},1\\ \frac{k+(a+b+1)/2}{m},\ldots,\frac{m-1+k+(a+b+1)/2}{m},\frac{2c+k}{m},\ldots,\frac{2c+m-1+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};1\right] = \frac{\sqrt{\pi}\,\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{a+b-1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a-1}{2}\right)\Gamma\left(c-\frac{b-1}{2}\right)} \tag{2.6}$$

It is worth noting that for m=1 results reduce to Eq. (1.3) to Eq. (1.9).
*Remark:* For other generalizations we refer to [12]- [15].

Our aim is to achieve a new class of double integral involving the generalization of classical summation theorems given by Masjed-Jamei and Eslahchi (2015), (2.2) to (2.8) using the following double integral due to Edwards (1954):

$$\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}\left(1-xy\right)^{1-\alpha-\beta}\,dxdy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.9}$$

## III. NEW CLASS OF DOUBLE INTEGRALS

**Theorem 1:** For $m\,\epsilon\mathbb{N}$ such that $m>0$ and $\Re(c-a-b)>0$ following result holds true:

$$\sum_{k=0}^{m-1}\frac{(a)_k}{k!}\int_0^1\int_0^1 y^{b+k}(1-x)^{b+k-1}(1-y)^{c-b-1}(1-xy)^{1-c-k}$$

$$\times{}_{m+1}F_m\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},1\\ \frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix}\;;\;\left(\frac{y(1-x)}{1-xy}\right)^m\right]dxdy$$

$$=\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}$$

**Proof:** For proving Theorem, denoting the right-hand side of the Eq. by $I$, expressing ${}_{m+1}F_m$ as a series,

$$I=\sum_{k=0}^{m-1}\frac{(a)_k}{k!}\int_0^1\int_0^1 y^{b+k}(1-x)^{b+k-1}(1-y)^{c-b-1}(1-xy)^{1-c-k}$$

$$\times\left(\frac{y^m(1-x)^m}{(1-xy)^m}\right)^n\sum_{n=0}^{\infty}\left(\frac{\left(\frac{a}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{a+m-1}{m}+\frac{k}{m}\right)_n(1)_n}{\left(\frac{1}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{m}{m}+\frac{k}{m}\right)_n n!}\right)dxdy,$$

changing the order of integration and summation

$$=\sum_{k=0}^{m-1}\frac{(a)_k}{k!}\sum_{n=0}^{\infty}\left(\frac{\left(\frac{a}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{a+m-1}{m}+\frac{k}{m}\right)_n(1)_n}{\left(\frac{1}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{m}{m}+\frac{k}{m}\right)_n n!}\right)$$

$$\times\int_0^1\int_0^1 y^{b+k+mn}(1-x)^{b+k+mn-1}(1-y)^{c-b-1}(1-xy)^{1-c-k-mn}\,dxdy,$$

and evaluating using double integral due to Edwards (1954), We have:

$$=\sum_{k=0}^{m-1}\frac{(a)_k}{k!}\sum_{n=0}^{\infty}\left(\frac{\left(\frac{a}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{a+m-1}{m}+\frac{k}{m}\right)_n(1)_n}{\left(\frac{1}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{m}{m}+\frac{k}{m}\right)_n n!}\right)$$
$$\times\frac{\Gamma(b+k+mn)\Gamma(c-b)}{\Gamma(c+k+mn)},$$

employing the formula Eq. (1.6) and

$$=\Gamma(c-b)\sum_{k=0}^{m-1}\frac{(a)_k}{k!}\sum_{n=0}^{\infty}\left(\frac{\left(\frac{a}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{a+m-1}{m}+\frac{k}{m}\right)_n(1)_n}{\left(\frac{1}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{m}{m}+\frac{k}{m}\right)_n n!}\right)$$
$$\times\frac{\Gamma(b+k)(b+k)_{mn}}{\Gamma(c+k)(c+k)_{mn}},$$

again, employing the formula given in Eq. (1.6) and following formula (Slater 1966):

$$(a)_{kn}=\left(\frac{a}{k}\right)_n\left(\frac{a}{k}+\frac{1}{k}\right)_n\left(\frac{a}{k}+\frac{2}{k}\right)_n\cdots\left(\frac{a}{k}+\frac{k-1}{k}\right)_n k^{kn}$$

the following outcome was attained:

$$=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{k=0}^{m-1}\frac{(a)_k(b)_k}{(c)_k k!}$$
$$\times\sum_{n=0}^{\infty}\left(\begin{array}{c}\frac{\left(\frac{a}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{a+m-1}{m}+\frac{k}{m}\right)_n(1)_n}{\left(\frac{1}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{m}{m}+\frac{k}{m}\right)_n n!}\\ \times\frac{\left(\frac{b}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{b+m-1}{m}+\frac{k}{m}\right)_n}{\left(\frac{c}{m}+\frac{k}{m}\right)_n\cdots\left(\frac{c+m-1}{m}+\frac{k}{m}\right)_n}\end{array}\right)$$

summing up the series

$$=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{k=0}^{m-1}\frac{(a)_k(b)_k}{(c)_k k!}$$
$$\times\,{}_{2m+1}F_{2m}\left[\begin{array}{c}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},\frac{b+k}{m},\ldots,\frac{b+m-1+k}{m},1\\ \frac{c+k}{m},\ldots,\frac{c+m-1+k}{m},\frac{1+k}{m},\ldots,\frac{m+k}{m}\end{array};\,1\right],$$

and applying the formula given in the Eq. (3.12), we reached to final conclusion given in Eq. (4.23)

By putting $m = 1, 2, 3$ in Eq. we subsequently get following new results:

$$\int_0^1\int_0^1 y^b(1-x)^{b-1}(1-y)^{c-b-1}(1-xy)^{1-c}\,{}_2F_1\left[\begin{matrix} a,1 \\ 1 \end{matrix}; \frac{y(1-x)}{1-xy}\right]dxdy = \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)},$$

$$\int_0^1\int_0^1 y^b(1-x)^{b-1}(1-y)^{c-b-1}(1-xy)^{1-c}\,{}_3F_2\left[\begin{matrix} \frac{a}{2},\frac{a}{2}+\frac{1}{2},1 \\ \frac{1}{2},1 \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^2\right]dxdy$$

$$+a\int_0^1\int_0^1 y^{b+1}(1-x)^{b}(1-y)^{c-b-1}(1-xy)^{-c}\,{}_3F_2\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\frac{a}{2}+1,1 \\ 1,\frac{3}{2} \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^2\right]dxdy$$

$$=\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}$$

and

$$\int_0^1\int_0^1 y^b(1-x)^{b-1}(1-y)^{c-b-1}(1-xy)^{1-c}\,{}_4F_3\left[\begin{matrix} \frac{a}{3},\frac{a}{3}+\frac{1}{3},\frac{a}{3}+\frac{2}{3},1 \\ \frac{1}{3},\frac{2}{3},1 \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^3\right]dxdy$$

$$+a\int_0^1\int_0^1 y^{b+1}(1-x)^{b}(1-y)^{c-b-1}(1-xy)^{-c}\,{}_4F_3\left[\begin{matrix} \frac{a}{3}+\frac{1}{3},\frac{a}{3}+\frac{2}{3},\frac{a}{3}+1,1 \\ \frac{2}{3},1,\frac{4}{3} \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^3\right]dxdy$$

$$+\frac{(a)_2}{2!}\int_0^1\int_0^1 y^{b+2}(1-x)^{b+1}(1-y)^{c-b-1}(1-xy)^{-(c+1)}$$

$$\times{}_4F_3\left[\begin{matrix} \frac{a}{3}+\frac{2}{3},\frac{a}{3}+1,\frac{a}{3}+\frac{4}{3},1 \\ 1,\frac{4}{3},\frac{5}{3} \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^3\right]dxdy$$

$$=\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}.$$

The following theorem can be obtained using a similar procedure. Hence, they are given without proof.

 

**Theorem 2:** For $m \in \mathbb{N}$ such that $m > 0$ and $\Re(b) > 0$ following result holds true:

$$\sum_{k=0}^{m-1} \frac{(a)_k}{k!} \int_0^1 \int_0^1 y^{b+k} (1-x)^{b+k-1} (1-y)^{a-2b} (1-xy)^{1-a-b-k} {}_{m+1}F_m \left[ \begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, 1 \\ \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix} ; \left( \frac{-y(1-x)}{1-xy} \right)^m \right] dxdy$$

$$= \frac{\Gamma(b)\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-2b)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}$$

For m= 1, 2, 3

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{c-b-1} (1-xy)^{1-a-b} {}_2F_1 \left[ \begin{matrix} a, 1 \\ 1 \end{matrix} ; -\frac{y(1-x)}{1-xy} \right] dxdy = \frac{\Gamma(b)\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-2b)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)},$$

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{a-2b} (1-xy)^{1-a-b} {}_3F_2 \left[ \begin{matrix} a/2, a/2 + 1/2, 1 \\ 1/2, 1 \end{matrix} ; \left( \frac{y(1-x)}{1-xy} \right)^2 \right] dxdy$$

$$-a \int_0^1 \int_0^1 y^{b+1} (1-x)^{b-1} (1-y)^{a-2b} (1-xy)^{1-a-b} {}_3F_2 \left[ \begin{matrix} a/2 + 1/2, a/2 + 1, 1 \\ 1, 3/2 \end{matrix} ; \left( \frac{y(1-x)}{1-xy} \right)^2 \right] dxdy$$

$$= \frac{\Gamma(b)\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-2b)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}$$

$$\int_0^1 \int_0^1 y^{b-1} (1-x)^{b-1} (1-y)^{a-2b} (1-xy)^{1-a-b} {}_4F_3 \left[ \begin{matrix} a/3, a/3 + 1/3, a/3 + 2/3, 1 \\ 1/3, 2/3, 1 \end{matrix} ; \left( -\frac{y(1-x)}{1-xy} \right)^3 \right] dxdy$$

$$-a \int_0^1 \int_0^1 y^b (1-x)^b (1-y)^{a-2b} (1-xy)^{-(a+b)} {}_4F_3 \left[ \begin{matrix} a/3 + 1/3, a/3 + 2/3, a/3 + 1, 1 \\ 2/3, 1, 4/3 \end{matrix} ; \left( -\frac{y(1-x)}{1-xy} \right)^3 \right] dxdy$$

$$+ \frac{(a)_2}{2!} \int_0^1 \int_0^1 y^{b+1} (1-x)^{b+1} (1-y)^{a-2b} (1-xy)^{-(a+b+1)} {}_4F_3 \left[ \begin{matrix} a/3 + 2/3, a/3 + 1, a/3 + 4/3, 1 \\ 4/3, 5/3, 1 \end{matrix} ; \left( -\frac{y(1-x)}{1-xy} \right)^3 \right] dxdy$$

$$= \frac{\Gamma(b)\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-2b)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}$$

**Theorem 3:** For $m \epsilon \mathbb{N}$ such that $m > 0$ following result holds true:

$$\sum_{k=0}^{m-1} \frac{(a)_k 2^{-k}}{k!} \int_0^1 \int_0^1 y^{b+k} (1-x)^{b+k-1} (1-y)^{\frac{1}{2}(a-b-1)} (1-xy)^{\frac{1}{2}(1-a-b)-k} {}_{m+1}F_m \left[ \begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, 1 \\ \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix} ; \left( \frac{y(1-x)}{2(1-xy)} \right)^m \right] dxdy$$

$$= \frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}(a-b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right) \Gamma\left(\frac{1}{2}(b+1)\right)}$$

For m = 1, 2, 3

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{\frac{1}{2}(a-b-1)} (1-xy)^{\frac{1}{2}(1-a-b)-k} {}_2F_1 \left[ \begin{matrix} a, 1 \\ 1 \end{matrix} ; \frac{y(1-x)}{2(1-xy)} \right] dxdy = \frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}(a-b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right) \Gamma\left(\frac{1}{2}(b+1)\right)},$$

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{\frac{1}{2}(a-b-1)} (1-xy)^{\frac{1}{2}(1-a-b)} {}_3F_2 \left[ \begin{matrix} a/2, a/2 + 1/2, 1 \\ 1/2, 1 \end{matrix} ; \left( \frac{y(1-x)}{2(1-xy)} \right)^2 \right] dxdy$$

$$+ \frac{a}{2} \int_0^1 \int_0^1 y^{b+1} (1-x)^b (1-y)^{a-2b} (1-xy)^{-\frac{1}{2}(1-a-b)} {}_3F_2 \left[ \begin{matrix} a/2 + 1/2, a/2 + 1, 1 \\ 1, 3/2 \end{matrix} ; \left( \frac{y(1-x)}{1-xy} \right)^2 \right] dxdy$$

$$= \frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}(a-b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right) \Gamma\left(\frac{1}{2}(b+1)\right)}$$

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{\frac{1}{2}(a-b-1)} (1-xy)^{\frac{1}{2}(1-a-b)} {}_4F_3 \left[ \begin{matrix} a/3, a/3 + 1/3, a/3 + 2/3, 1 \\ 1/3, 2/3, 1 \end{matrix} ; \left( \frac{y(1-x)}{2(1-xy)} \right)^3 \right] dxdy$$

$$+ \frac{a}{2} \int_0^1 \int_0^1 y^{b+1} (1-x)^b (1-y)^{\frac{1}{2}(a-b-1)} (1-xy)^{-\frac{1}{2}(1-a-b)} {}_4F_3 \left[ \begin{matrix} a/3 + 1/3, a/3 + 2/3, a/3 + 1, 1 \\ 2/3, 1, 4/3 \end{matrix} ; \left( \frac{y(1-x)}{2(1-xy)} \right)^3 \right] dxdy$$

$$+ \frac{(a)_2}{2^2 2!} \int_0^1 \int_0^1 y^{b+1} (1-x)^{b+1} (1-y)^{\frac{1}{2}(a-b-1)} (1-xy)^{-\frac{1}{2}(3-a-b)} {}_4F_3 \left[ \begin{matrix} a/3 + 2/3, a/3 + 1, a/3 + 4/3, 1 \\ 4/3, 5/3, 1 \end{matrix} ; \left( \frac{y(1-x)}{2(1-xy)} \right)^3 \right] dxdy$$

$$= \frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}(a-b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right) \Gamma\left(\frac{1}{2}(b+1)\right)}$$

**Theorem 4**: For $m \in \mathbb{N}$ such that $m > 0$ following result holds true:

$$\sum_{k=0}^{m-1}\frac{(a)_k 2^{-k}}{k!}\int_0^1\int_0^1 y^{1-a+k}(1-x)^{k-a}(1-y)^{b+a-2}(1-xy)^{1-b-k}\,{}_{m+1}F_m\left[\begin{matrix}\frac{a+k}{m},\ldots,\frac{a+m-1+k}{m},1\\ \frac{1+k}{m},\ldots,\frac{m+k}{m}\end{matrix};\left(\frac{y(1-x)}{2(1-xy)}\right)^m\right]dxdy$$

$$=\frac{\sqrt{\pi}\,\Gamma(1-a)\Gamma(b+a-1)}{2^{b-1}\Gamma\left(\frac{b}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}$$

For m= 1, 2, 3

$$\int_0^1\int_0^1 y^{1-a}(1-x)^{-a}(1-y)^{b+a-2}(1-xy)^{1-b}\,{}_2F_1\left[\begin{matrix}a,1\\ 1\end{matrix};\frac{y(1-x)}{2(1-xy)}\right]dxdy=\frac{\sqrt{\pi}\,\Gamma(1-a)\Gamma(b+a-1)}{2^{b-1}\Gamma\left(\frac{b}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)},$$

$$\int_0^1\int_0^1 y^{1-a}(1-x)^{-a}(1-y)^{b+a-2}(1-xy)^{1-b}\,{}_3F_2\left[\begin{matrix}a/2,a/2+1/2,1\\ 1/2,1\end{matrix};\left(\frac{y(1-x)}{2(1-xy)}\right)^2\right]dxdy$$

$$+\frac{a}{2}\int_0^1\int_0^1 y^{2-a}(1-x)^{1-a}(1-y)^{b+a-2}(1-xy)^{-b}\,{}_3F_2\left[\begin{matrix}a/2+1/2,a/2+1,1\\ 1,3/2\end{matrix};\left(\frac{y(1-x)}{2(1-xy)}\right)^2\right]dxdy$$

$$=\frac{\sqrt{\pi}\,\Gamma(1-a)\Gamma(b+a-1)}{2^{b-1}\Gamma\left(\frac{b}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}$$

$$\int_0^1\int_0^1 y^{1-a}(1-x)^{-a}(1-y)^{b+a-2}(1-xy)^{1-b}\,{}_4F_3\left[\begin{matrix}a/3,a/3+1/3,a/3+2/3,1\\ 1/3,2/3,1\end{matrix};\left(\frac{y(1-x)}{2(1-xy)}\right)^3\right]dxdy$$

$$+\frac{a}{2}\int_0^1\int_0^1 y^{2-a}(1-x)^{1-a}(1-y)^{b+a-2}(1-xy)^{-b}\,{}_4F_3\left[\begin{matrix}a/3+1/3,a/3+2/3,a/3+1,1\\ 2/3,1,4/3\end{matrix};\left(\frac{y(1-x)}{2(1-xy)}\right)^3\right]dxdy$$

$$+\frac{(a)_2}{2^{-2}2!}\int_0^1\int_0^1 y^{3-a}(1-x)^{2-a}(1-y)^{b+a-2}(1-xy)^{-(b+1)}\,{}_4F_3\left[\begin{matrix}a/3+2/3,a/3+1,a/3+4/3,1\\ 4/3,5/3,1\end{matrix};\left(\frac{y(1-x)}{2(1-xy)}\right)^3\right]dxdy$$

$$=\frac{\sqrt{\pi}\,\Gamma(1-a)\Gamma(b+a-1)}{2^{b-1}\Gamma\left(\frac{b}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}$$

**Theorem 5**: For $m \epsilon \mathbb{N}$ such that $m > 0$ following result holds true:

$$\sum_{k=0}^{m-1} \frac{(a)_k (c)_k}{\left(\frac{1}{2}(a+b+1)\right)_k k!} \int_0^1 \int_0^1 y^{b+k} (1-x)^{b+k-1} (1-y)^{2c-b-1} (1-xy)^{1-2c-k}$$

$$\times {}_{2m+1}F_{2m} \left[ \begin{matrix} \frac{a+k}{m}, \ldots, \frac{a+m-1+k}{m}, \frac{c+k}{m}, \ldots, \frac{c+m-1+k}{m}, 1 \\ \frac{((a+b+1)/2)+k}{m}, \ldots, \frac{((a+b-1)/2)+k+m}{m}, \frac{1+k}{m}, \ldots, \frac{m+k}{m} \end{matrix} ; \left(\frac{y(1-x)}{1-xy}\right)^m \right] dxdy$$

$$= \frac{\pi\, \Gamma(b)\Gamma(2c-b)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma(c)\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(c+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)}$$

For m= 1, 2, 3

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{c-b-1} (1-xy)^{1-2c} {}_3F_2 \left[ \begin{matrix} a, c, 1 \\ \frac{1}{2}(a+b+1), 1 \end{matrix} ; \frac{y(1-x)}{1-xy} \right] dxdy$$

$$= \frac{\pi\, \Gamma(b)\Gamma(2c-b)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma(c)\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(c+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)},$$

$$\int_0^1 \int_0^1 y^b (1-x)^{b-1} (1-y)^{2c-b-1} (1-xy)^{1-2c} {}_5F_4 \left[ \begin{matrix} a/2, a/2 + 1/2, c/2, c/2 + 1/2, 1 \\ \frac{1/2(a+b+1)}{2}, \frac{1/2(a+b+1)}{2} + 1/2, 1/2, 1 \end{matrix} ; \left(\frac{y(1-x)}{1-xy}\right)^2 \right] dxdy$$

$$+ \frac{ac}{\frac{1}{2}(a+b+1)} \int_0^1 \int_0^1 y^{b+1} (1-x)^b (1-y)^{2c-b-1} (1-xy)^{-2c}$$

$$\times {}_5F_4 \left[ \begin{matrix} a/2 + 1/2, a/2 + 1, c/2 + 1/2, c/2 + 1, 1 \\ \frac{1/2(a+b+1)}{2} + 1/2, \frac{1/2(a+b+1)}{2} + 1, 1, 3/2 \end{matrix} ; \left(\frac{y(1-x)}{1-xy}\right)^2 \right] dxdy$$

$$= \frac{\pi\, \Gamma(b)\Gamma(2c-b)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma(c)\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(c+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)}$$

$$\int_0^1\int_0^1 y^b(1-x)^{b-1}(1-y)^{2c-b-1}(1-xy)^{1-2c-k}$$

$$\times{}_7F_6\left[\begin{matrix} a/3, a/3+1/3, a/3+2/3, c/3, c/3+1/3, c/3+2/3, 1 \\ \frac{1/2(a+b+1)}{3}, \frac{1/2(a+b+1)}{3}+\frac{1}{3}, \frac{1/2(a+b+1)}{3}+\frac{2}{3}, 1/3, 2/3, 1 \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^3\right]dxdy$$

$$+\frac{ac}{\frac{1}{2}(a+b+1)}\int_0^1\int_0^1 y^{b+1}(1-x)^b(1-y)^{2c-b-1}(1-xy)^{-2c}$$

$$\times{}_7F_6\left[\begin{matrix} a/3+1/3, a/3+2/3, a/3+1, c/3+1/3, c/3+2/3, c/3+1, 1 \\ \frac{1/2(a+b+1)}{3}+\frac{1}{3}, \frac{1/2(a+b+1)}{3}+\frac{2}{3}, \frac{1/2(a+b+1)}{3}, +1, 2/3, 1, 4/3 \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^3\right]dxdy$$

$$+\frac{(a)_2(c)_2}{\left(\frac{1}{2}(a+b+1)\right)_2 2!}\int_0^1\int_0^1 y^{b+2}(1-x)^{b+1}(1-y)^{2c-b-1}(1-xy)^{-(2c+1)}$$

$$\times{}_7F_6\left[\begin{matrix} a/3+2/3, a/3+1, a/3+4/3, a/3+2/3, a/3+1, a/3+4/3, 1 \\ \frac{1/2(a+b+1)}{3}+\frac{2}{3}, \frac{1/2(a+b+1)}{3}+1, \frac{1/2(a+b+1)}{3}+\frac{4}{3}, 4/3, 5/3, 1 \end{matrix}; \left(\frac{y(1-x)}{1-xy}\right)^3\right]dxdy$$

$$=\frac{\pi\,\Gamma(b)\Gamma(2c-b)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma(c)\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(c+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)}$$

## IV. CONCLUSION

Using generalized hypergeometric functions, we have developed novel and intriguing conclusions in this study that pertain to double integrals. This was accomplished by using the Masjed-Jamei and Eslahchi generalization of the traditional summation theorem. These formulas can be distinguished as master formulas due to the presence of m, from which a significant number of intriguing formulas and outcomes can be obtained.

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