## BS-ALGEBRAS AND ITS FUZZY IDEAL

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**Abstract:** In this research article, a generalization of B-Algebras called BS-Algebras is introduced. We include fuzzy ideal in BS-Algebras. Some new characterizations were given.

**Keywords:** BS-Algebras, Fuzzy ideal, Homomorphism, Cartesian Product.

# **1. Introduction**

Fuzzy subsets was introduced by L.A.Zadeh[4], many research people investigated the generalization of the notion of fuzzy subset.In 1966, Imai and Iseki established two classes of abstract algebras, they are BCK-algebras and BCI-algebras[2]. J.Neggers and H.S. Kim brought the notion of B-algebras[3] which is a generalisation of BCK-algebras. As an extension, the author newly initiate the notion of BS-algebras, as a generalisation of B-algebras. In this article, the author study the concepts of BS-Algebras with its examples. The author apply the concept of fuzzy ideal in BS-Algebras and find some of their basic properties and explore some algebraic nature of fuzzy ideal in BS-algebras. The homomorphic behaviour of fuzzy ideal of BS-algebras have been investigated. Finally, Cartesian product is also applied in fuzzy ideal of BS-algebras.

# **2. Preliminaries**

**Definition 2.1**. *[3]* A B-algebra *𝕭≠𝜙* with a constant 0 and a binary operation ∗ satisfying the following conditions

(i) 𝛼 ∗ 𝛼 = 0

(ii) *𝛼* ∗ 0 = *𝛼*

(iii) (*𝛼* ∗ *𝛽*) ∗ *𝛾* = *𝛼* ∗ (*𝛾* ∗ (0 ∗ *𝛽*)) for all *𝛼, 𝛽, 𝛾* ∈ *𝕭*

**Definition 2.2.** If *𝛗*1 and *𝛗*2 be any two fuzzy sets of *𝕭*. Then its Intersection is defined by *𝛗*1 ∩ *𝛗*2 = *min*{*𝛗*1(*𝛼*)*, 𝛗*2(*𝛼*)} for all *𝛼* ∈ *𝕭*

**Definition 2.3**. Let *A* = {*𝛗*1(*𝛼*)*, 𝛼* ∈ *𝕭*} and *B* = {*𝛗*2(*𝛼*)*, 𝛼* ∈ *𝕭*} be any two fuzzy sets on *𝕭*. Then the cartesian product *A* × *B* = {*𝛗*1 × *𝛗*2(*𝛼, 𝛽*) : *𝛼, 𝛽* ∈ *𝕭*} which is defined by (*𝛗*1 × *𝛗*2)(*𝛼, 𝛽*) = *min*{*𝛗*1(*𝛼*)*, 𝛗*2(*𝛽*)} where *𝛗*1 × *𝛗*2 : *𝕭* × *𝕭* → [0*,* 1] for all *𝛼, 𝛽* ∈ *𝕭*.

**Definition 2.4**. *[1]* Let *𝛗 be a fuzzy set* *𝕭* is called the doubt fuzzy bi-ideal of *𝕭* if

i) *𝛗*(1) ≤ *𝛗*(*𝛼*)

ii) *𝛗*(*𝛽* ∗ *𝛾*) ≤ *max*{*𝛗*(*𝛼*)*, 𝛗*(*𝛼* ∗ (*𝛽* ∗ *𝛾*))} for all *𝛼, 𝛽, 𝛾* ∈ *𝕭*

# **3. BS-Algebras (***𝕭***)**

**Definition 3.1**. A *𝕭≠𝜙* with a constant 1 and a binary operation ∗ is called a BS-Algebras which satisfies the conditionsof the following

(i) 𝛼 ∗ 𝛼 = 1

(ii) 𝛼 ∗ 1 = 𝛼

(iii) 1 ∗ *𝛼* = *𝛼*

(iv) (*𝛼* ∗ *𝛽*) ∗ *𝛾* = *𝛼* ∗ (*𝛾* ∗ (1 ∗ *𝛽*)) for all *𝛼, 𝛽, 𝛾* ∈ *𝕭*

A binary relation ≤ on *𝕭* can be defined by *𝛼* ≤ *𝛽* iff *𝛼* ∗ *𝛽* = 1

**Example 3.2**. i) A set *𝕭* = {1*, a, b, c*} which satisfies the table below

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | 1 | a | b | c |
| 1 | 1 | a | b | c |
| a | a | 1 | c | b |
| b | b | c | 1 | a |
| c | c | b | a | 1 |

Then (*𝕭,* ∗*,* 1) is a BS-algebra.

1. A set *𝕭* = {1*, a, b*} which satisfies the table below

|  |  |  |  |
| --- | --- | --- | --- |
| \* | 1 | a | b |
| 1 | 1 | a | b |
| a | a | 1 | a |
| b | b | a | 1 |

Then (*𝕭,* ∗*,* 1) is a BS-algebra.

(iii) A set *𝕭* = {1*, a, b, c, d, e*} which satisfies the table below

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| \* | 1 | a | b | c | d | e |
| 1 | 1 | a | b | c | d | e |
| a | a | 1 | b | d | e | c |
| b | b | a | 1 | e | c | d |
| c | c | d | e | 1 | b | a |
| d | d | e | c | a | 1 | b |
| e | e | c | d | b | a | 1 |

If we put *𝛽* = *𝛼* in (*𝛼* ∗ *𝛽*) ∗ *𝛾* = *𝛼* ∗ (*𝛾* ∗ (1 ∗ *𝛽*)), then we have (*𝛼* ∗ *𝛼*) ∗ *𝛾* = *𝛼* ∗ (*𝛾* ∗ (1 ∗ *𝛼*)) → (I)

⇒ 1 ∗ *𝛾* = *𝛼* ∗ (*𝛾* ∗ (1 ∗ *𝛼*))

If we put *𝛾* = *𝛼* in (I) then we get 1 ∗ *𝛼* = *𝛼* ∗ (*𝛼* ∗ (1 ∗ *𝛼*)) → (*II*) Using (i) and (I) and *𝛾* = 1 it follows that

1 = *𝛼* ∗ (1 ∗ (1 ∗ *𝛼*))*,* → (*III*)

we see that the four conditions (i),(ii),(iii) and (iv) are independent.

* 1. A set *𝕭* = {1*, a, b*} which satisfies the table below

|  |  |  |  |
| --- | --- | --- | --- |
| \* | 1 | a | b |
| 1 | 1 | a | 1 |
| a | a | 1 | a |
| b | 1 | a | 1 |

Then the conditions (i) and (iv) hold, but (ii) and (iii) does not hold

since *b* ∗ 1 = 1 ≠ *b*

* 1. The set *𝕭* = {0*,* 1*,* 2} which satisfies the table below

|  |  |  |  |
| --- | --- | --- | --- |
| \* | 1 | a | b |
| 1 | 1 | a | b |
| a | a | a | a |
| b | b | a | b |

The axioms (ii), (iii) and (iv) satisfies

but it does not hold (i) since *a* ∗ *a* = *a≠* 1 and *b* ∗ *b* = *b* ≠ 1

* 1. Let *𝕭* = {1*, a, b, c*} be a set which satisfies the table below

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | 1 | a | b | c |
| 1 | 1 | a | b | c |
| a | a | 1 | 1 | 1 |
| b | b | 1 | 1 | a |
| c | c | 1 | 1 | 1 |

Then (*𝕭,* ∗*,* 1) satisfies the conditions (i), (ii) and (iii) but it does not hold (iv) since (*b* ∗ *c*) ∗ 1 = *a* ≠ *b* = *b* ∗ (1 ∗ (1 ∗ *c*))

**Theorem 3.3**. If (𝕭, ∗, 1) is a BS-algebras, then prove that 𝛽 ∗ 𝛾 = 𝛽 ∗ (1 ∗ (1 ∗ 𝛾)) for all 𝛽, 𝛾 ∈ 𝕭

**Proof*.***This comes from the condition *𝛼* ∗ 1 = *𝛼* and (*𝛼* ∗ *𝛽*) ∗ *𝛾* = *𝛼* ∗ (*𝛾* ∗ (1 ∗ *𝛽*)) Now, *𝛽* ∗ *𝛾* = (*𝛽* ∗ *𝛾*) ∗ 1 (by (ii))

= *𝛽* ∗ (1 ∗ (1 ∗ *𝛾*)) (by (iv))

**Theorem 3.4**. If (𝕭, ∗, 1) is a BS-algebra, then prove that (𝛼 ∗ 𝛽) ∗ (1 ∗ 𝛽) = 𝛼

for all 𝛼, 𝛽 ∈ 𝕭

**Proof.**From (iv) with *𝛾* = 1 ∗ *𝛽* we have (*𝛼* ∗ *𝛽*) ∗ (1 ∗ *𝛽*) = *𝛼* ∗ ((1 ∗ *𝛽*) ∗ (1 ∗ *𝛽*)) From condition (i) (*𝛼* ∗ *𝛽*) ∗ (1 ∗ *𝛽*) = *𝛼* ∗ 1

From condition (ii),it becomes (*𝛼* ∗ *𝛽*) ∗ (1 ∗ *𝛽*) = *𝛼*

**Theorem 3.5.** If (𝕭, ∗, 1) is a BS-algebra, then prove that 𝛼 ∗ 𝛾 = 𝛽 ∗ 𝛾 ⇒ 𝛼 = 𝛽 for all 𝛼, 𝛽, 𝛾 ∈ 𝕭

**Proof.**If *𝛼* ∗ *𝛾* = *𝛽* ∗ *𝛾*, then (*𝛼* ∗ *𝛾*) ∗ (1 ∗ *𝛾*) = (*𝛽* ∗ *𝛾*) ∗ (1 ∗ *𝛾*)

and by previous theorem, we get *𝛼* = *𝛽*

**Theorem 3.6.**  If (𝕭, ∗, 1) is a BS-algebras, then prove that

𝛼 ∗ (𝛽 ∗ 𝛾) = (𝛼 ∗ (1 ∗ 𝛾)) ∗ 𝛽 for all 𝛼, 𝛽, 𝛾 ∈ 𝕭

**Proof**. Using (iv) we obtain (*𝛼* ∗ (1 ∗ *𝛾*)) ∗ *𝛽* = *𝛼* ∗ (*𝛽* ∗ (1 ∗ (1 ∗ *𝛾*)))

= *𝛼* ∗ (*𝛽* ∗ *𝛾*) (by thm(3.3))

**Theorem 3.7**. Let (𝕭, ∗, 1) be a BS-algebra.Then prove that for all 𝛼, 𝛽 ∈ 𝕭

1. 𝛼 ∗ 𝛽 = 1 ⇒ 𝛼 = 𝛽

(ii)1 ∗ 𝛼 = 1 ∗ 𝛽 ⇒ 𝛼 =𝛽 (iii)1 ∗ (1 ∗ 𝛼) = 𝛼

**Proof**. (i) Since *𝛼* ∗ *𝛽* = 1 ⇒ *𝛼* ∗ *𝛽* = *𝛽* ∗ *𝛽*, by theorem (3.5), we get *𝛼* = *𝛽*

1. If 1 ∗ *𝛼* = 1 ∗ *𝛽*, then 1 = *𝛼* ∗ *𝛼* = (*𝛼* ∗ *𝛼*) ∗ 1

= *𝛼* ∗ (1 ∗ (1 ∗ *𝛼*))

= *𝛼* ∗ (1 ∗ (1 ∗ *𝛽*))

= (*𝛼* ∗ *𝛽*) ∗ 1

1 = *𝛼* ∗ *𝛽*

By (i), *𝛼* = *𝛽*

1. For all *𝛼* ∈ *𝕭*, we get 1 ∗ *𝛼* = (1 ∗ *𝛼*) ∗ 1 (*b𝛽* (*ii*))

= 1 ∗ (1 ∗ (1 ∗ *𝛼*)) (*b𝛽* (*iv*))

By (ii) part of this theorem, we have *𝛼* = 1 ∗ (1 ∗ *𝛼*)

**Theorem 3.8.** If (𝕭, ∗, 1) is a BS-algebra, then prove that (𝛼 ∗ 𝛽) ∗ 𝛽 = 𝛼 ∗ 𝛽2

for all 𝛼, 𝛽 ∈ 𝕭

**Proof**.From (iv), We have (*𝛼* ∗ *𝛽*) ∗ *𝛽* = *𝛼* ∗ (*𝛽* ∗ (1 ∗ *𝛽*))

= *𝛼* ∗ (*𝛽* ∗ *𝛽*)

= *𝛼* ∗ *𝛽* 2

**Theorem 3.9.** If (𝕭, ∗, 1) is a BS-algebra, then prove that (1 ∗ 𝛽) ∗ (𝛼 ∗ 𝛽) = 𝛼

for all 𝛼, 𝛽 ∈ 𝕭

**Proof.** From theorem (3.6), (1 ∗ *𝛽*) ∗ (*𝛼* ∗ *𝛽*) = ((1 ∗ *𝛽*) ∗ (1 ∗ *𝛽*)) ∗ *𝛼*

= 1 ∗ *𝛼*

= *𝛼*

**Definition 3.10.** A BS-algebra (*𝕭,* ∗*,* 1) is said to be commutative if

*𝛼* ∗ (1 ∗ *𝛽*) = *𝛽* ∗ (1 ∗ *𝛼*) for all *𝛼, 𝛽* ∈ *𝕭*

**Note 3.11**. The BS-algebra in example:3.2 (i) is commutative but the algebra in example:3.2 (iii)is not commutative since

*c* ∗ (1 ∗ *d*) = *b≠a* = *d* ∗ (1 ∗ *c*)

**Theorem 3.12.** If (𝕭, ∗, 1) is commutative, then prove that (1 ∗ 𝛼) ∗ (1 ∗ 𝛽) = 𝛽 ∗ 𝛼

for all 𝛼, 𝛽 ∈ 𝕭

**Proof.**Since (*𝕭,* ∗*,* 1) is commutative, then (1 ∗ *𝛼*) ∗ (1 ∗ *𝛽*) = *𝛽* ∗ (1 ∗ (1 ∗ *𝛼*))

= *𝛽* ∗ *𝛼* (by thm(3.3))

**Theorem 3.13.** If (𝕭, ∗, 1) is commutative, then prove that 𝛼 ∗ (𝛼 ∗ 𝛽) = 𝛽 for all

𝛼, 𝛽 ∈ 𝕭

**Proof.** By theorem (3.6),

Now*, 𝛼* ∗ (*𝛼* ∗ *𝛽*) = (*𝛼* ∗ (1 ∗ *𝛽*)) ∗ *𝛼*

= (*𝛽* ∗ (1 ∗ *𝛼*)) ∗ *𝛼* (since (*𝕭,* ∗*,* 1) is commutative)

= *𝛽* ∗ (*𝛼* ∗ *𝛼*)

= *𝛽* ∗ 1

= *𝛽*

**Corollary 3.14.** If (𝕭, ∗, 1) is commutative, then the left cancellation law holds

(i.e) (𝛼 ∗ 𝛽) = 𝛼 ∗ 𝛽′ ⟹ 𝛽 = 𝛽′

**Proof*.***From theorem 3.13, we have *𝛽* = *𝛼* ∗ (*𝛼* ∗ *𝛽*) = *𝛼* ∗ (*𝛼* ∗ *𝛽*′) = *𝛽*′

**Theorem 3.15.** If (𝕭, ∗, 1) is commutative, then prove that (1 ∗ 𝛼) ∗ (𝛼 ∗ 𝛽) = 𝛽 ∗ 𝛼2

for all 𝛼, 𝛽 ∈ 𝕭

**Proof.**

Now, (1 ∗ *𝛼*) ∗ (*𝛼* ∗ *𝛽*) = ((1 ∗ *𝛼*) ∗ (1 ∗ *𝛽*)) ∗ *𝛼* (by thm (3.6))

= (*𝛽* ∗ *𝛼*) ∗ *𝛼* (by thm (3.12))

= *𝛽* ∗ *𝛼*2 (by thm (3.8))

# **4. Fuzzy Ideal**

**Definition 4.1**. Let A be a fuzzy set in BS-Algebra 𝕭 is called a fuzzy ideal of 𝕭 if it satisfies the conditions below

i) 𝛗(1) ≥ 𝛗(𝛽)

ii) 𝛗(𝛽) ≥ min{𝛗(𝛽 ∗ 𝛼), 𝛗(𝛼)}for all 𝛼,𝛽 ∈ 𝕭

**Example 4.2**. A set *𝕭* = {1*, a, b, c*} which satisfies the table below

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | 1 | a | b | c |
| 1 | 1 | a | b | c |
| a | a | 1 | c | b |
| b | b | c | 1 | a |
| c | c | b | a | 1 |

Then (*𝕭,* ∗*,* 1) is a BS-algebra. Define a fuzzy set *𝛗* in *𝕭* by *𝛗*(1) = *𝛗*(*b*) = 0*.*7 and

*𝛗*(*a*) =*𝛗*(*c*) = 0*.*3 is the fuzzy ideal of *𝕭*.

**Theorem 4.3.** Let 𝛗 be a fuzzy ideal of 𝕭 is a, then

for all 𝛼 ∈ 𝕭, 𝛗(1) ≥ 𝛗(𝛼)

**Proof.** Straight forward

**Theorem 4.4.** If 𝛗1 and 𝛗2 be two fuzzy ideals of 𝕭, then 𝛗1 ∩ 𝛗2 is also a fuzzy ideal of 𝕭.

**Proof.**

Now, (*𝛗*1 ∩ *𝛗*2)(1) = (*𝛗*1 ∩ *𝛗*2)(*𝛼* ∗ *𝛼*)(b𝛽 (i))

≥ *min*{(*𝛗*1 ∩ *𝛗*2)(*𝛼*)*,* (*𝛗*1 ∩ *𝛗*2)(*𝛼*)}

= (*𝛗*1 ∩ *𝛗*2)(*𝛼*)

Therefore, (*𝛗*1 ∩ *𝛗*2)(1) ≥ (*𝛗*1 ∩ *𝛗*2)(*𝛼*)

Also, (*𝛗*1 ∩ *𝛗*2)(*𝛽*) = *min*{*𝛗*1(*𝛽*)*, 𝛗*2(*𝛽*)}

≥ *min*{*min*{*𝛗*1(*𝛼*)*, 𝛗*1(*𝛽*∗ *𝛼*)}*, min*{*𝛗*2(*𝛼*)*, 𝛗*2(*𝛽* ∗ *𝛼*)}}

= *min*{*min*{*𝛗*1(*𝛼*)*, 𝛗*2(*𝛼*)}*, min*{*𝛗*1(*𝛽* ∗ *𝛼*)*, 𝛗*2(*𝛽* ∗ *𝛼*)}}

= *min*{(*𝛗*1 ∩ *𝛗*2)(*𝛼*)*,* (*𝛗*1 ∩ *𝛗*2)(*𝛽* ∗ *𝛼*)}

(*𝛗*1 ∩ *𝛗*2)(*𝛽*) ≥ *min*{(*𝛗*1 ∩ *𝛗*2)(*𝛼*)*,* (*𝛗*1 ∩ *𝛗*2)(*𝛽* ∗ *𝛼*)}

Hence *𝛗*1 ∩ *𝛗*2 is a fuzzy ideal of *𝕭*.

**Theorem 4.5.** Let 𝛗 be a fuzzy ideals of 𝕭. If 𝛼 ∗ 𝛽 ≤ 𝛾,

then 𝛗(𝛼) ≥ min{𝛗(𝛽), 𝛗(𝛾)}

**Proof.**Let *𝛼, 𝛽, 𝛾* ∈ *𝕭* such that *𝛼* ∗ *𝛽* ≤ *𝛾*. Then (*𝛼* ∗ *𝛽*) ∗ *𝛾* = 1,

*𝛗*(*𝛼*) ≥ *min*{*𝛗*(*𝛼* ∗ *𝛽*)*, 𝛗*(*𝛽*)}

≥ *min*{*min*{*𝛗*((*𝛼* ∗ *𝛽*) ∗ *𝛾*)*, 𝛗*(*𝛾*)}*, 𝛗*(*𝛽*)}

= *min*{*min*{*𝛗*(1)*, 𝛗*(*𝛾*)}*, 𝛗*(*𝛽*)}

= *min*{*𝛗*(*𝛾*)*, 𝛗*(*𝛽*)}

Therefore, *𝛗*(*𝛼*) ≥ *min*{*𝛗*(*𝛾*)*, 𝛗*(*𝛽*)}

**Theorem 4.6.** Let 𝛗 be a fuzzy ideals of a BS-algebras 𝕭. If 𝛼 ≤ 𝛽, then 𝛗(𝛼) ≥ 𝛗(𝛽) (i.e) order reversing

**Proof.** Let *𝛼, 𝛽* ∈ *𝕭* such that *𝛼* ≤ *𝛽*. Then *𝛼* ∗ *𝛽* = 1

*𝛗*(*𝛼*) ≥ *min*{*𝛗*(*𝛼* ∗ *𝛽*)*, 𝛗*(*𝛽*)}(by (ii) in def 4.1)

= *min*{*𝛗*(1)*, 𝛗*(*𝛽*)}

= *𝛗*(*𝛽*) *𝛗*(*𝛼*) ≥ *𝛗*(*𝛽*)

Hence, *𝛗* is order reversing.

**Theorem 4.7.** Let B be a crisp subset of BS-algebras 𝕭. Suppose a fuzzy set F = 𝛗(𝛼) in 𝕭 defined by 𝛗(𝛼) = λ if 𝛼 ∈ Y and 𝛗(𝛼) = τ if 𝛼 ∉ Y for all λ, τ ∈ [0, 1] with λ ≥ τ. Then F is a fuzzy ideal of 𝕭 iff Y is a ideal of 𝕭

**Proof.**Assume that Fis a fuzzy ideal of *𝕭*. Let *𝛼* ∈ Y.

Let *𝛼, 𝛽* ∈ *𝕭* be such that *𝛽* ∗ *𝛼* ∈ Yand *𝛼* ∈ Y. Then *𝛗*(*𝛽* ∗ *𝛼*) = *λ* = *𝛗*(*𝛼*),

and hence *𝛗*(*𝛽*) ≥ *min*{*𝛗*(*𝛼*)*, 𝛗*(*𝛽* ∗ *𝛼*)} = *λ.* Thus *𝛗*(*𝛽*) = *λ* (i.e) *𝛽* ∈ Y*.*

Therefore Yis a ideal of *𝕭*.

Conversely, assume that Yis an ideal of *𝕭*. Let *𝛽* ∈*𝕭.*

Let *𝛼, 𝛽* ∈ *𝕭*. If *𝛽* ∗ *𝛼* ∈ Yand *𝛼* ∈ Y, then *𝛽* ∈ Y*.*

Hence, *𝛗*(*𝛽*) = *λ* = *min*{*𝛗*(*𝛽* ∗ *𝛼*)*, 𝛗*(*𝛼*)}

If *𝛽* ∗ *𝛼 ∉ Y* and *𝛼 ∉ Y*, then clearly *𝛗*(*𝛽*) ≥ *min*{*𝛗*(*𝛼*)*, 𝛗*(*𝛽* ∗ *𝛼*)}

If exactly one of *𝛽* ∗ *𝛼* and *𝛼* belong to Y, then exactly one of *𝛗*(*𝛽* ∗ *𝛼*)*, 𝛗*(*𝛼*) is equal to *τ* . Therefore, *𝛗*(*𝛽*) ≥ *τ* = *min*{*𝛗*(*𝛼*)*, 𝛗*(*𝛽* ∗ *𝛼*)}

Consequently, *A* is a fuzzy ideal of  *𝕭*.

**Theorem 4.8.** Let 𝛗 be an ideal of 𝕭 then the set 𝞾 (𝛗 : x) is an ideal of 𝕭 for every x ∈ [0, 1]

**Proof.** Suppose that *𝛗* is an fuzzy ideal of *𝕭*. For x∈ [0*,* 1]. Let *𝛼, 𝛽* ∈ *𝕭* be such that *𝛽* ∗ *𝛼* ∈ 𝞾(*𝛗* : x) and *𝛼* ∈ 𝞾(*𝛗* : x). Then *𝛗*(*𝛽*) ≥ *min*{*𝛗*(*𝛼*)*, 𝛗*(*𝛽* ∗ *𝛼*)}.

Then *𝛽* ∈ 𝞾(*𝛗* : x). Hence 𝞾(*𝛗* : x) is an ideal of *𝕭*.

**Definition 4.9.** Let f : 𝕭 →𝕭’ be the two BS-algebras. Let Y be a fuzzy set in 𝕭’ . Then the inverse image of Y is defined as

f −1(𝛗)(𝛼) = 𝛗(f (𝛼)). The set *f* −1(*B*) = {*f* −1(*𝛗*)(*𝛼*) : *𝛼* ∈ *𝕭’*} *is a fuzzy set.*

**Theorem 4.10.** Let f : 𝕭 →𝕭’ be a homomorphism of BS-algebras. If Y is a fuzzy ideal of 𝕭’, then the pre-image f −1(Y) in 𝕭’ is a fuzzy ideal of 𝕭

**Proof**. For all *𝛼* ∈ *𝕭’*, *f* −1(*𝛗*)(*𝛼*) = *𝛗*(*f* (*𝛼*)) ≤ *𝛗*(1) = *𝛗*(*f* (1)) = *f* −1(*𝛗*)(1) Therefore, *f* −1(*𝛗*)(*𝛼*) ≤ *f* −1(*𝛗*)(1)

Let *𝛼, 𝛽* ∈ *𝕭’*. Then *f* −1(*𝛗*)(*𝛼*) = *𝛗*(*f* (*𝛼*))

≥ *min*{*𝛗*(*f* (*𝛼*) ∗ *f* (*𝛽*))*, 𝛗*(*f* (*𝛽*))}

≥ *min*{*𝛗*(*f* (*𝛼* ∗ *𝛽*))*, 𝛗*(*f* (*𝛽*))}

= *min*{*f* −1(*𝛗*)(*𝛼* ∗ *𝛽*)*, f* −1(*𝛗*)(*𝛽*)}

Therefore, *f* −1(*𝛗*)(*𝛼*) ≥ *min*{*f* −1(*𝛗*)(*𝛼* ∗ *𝛽*)*, f* −1(*𝛗*)(*𝛽*)}

Hence *f* −1(Y) = {*f* −1(*𝛗*)(*𝛼*) : *𝛼* ∈ *𝕭’*} is a fuzzy ideal of *𝕭*

**Theorem 4.11**. Let f : 𝕭→𝕭’ be an onto homomorphism of BS-algebra. Then Y is a fuzzy ideal of 𝕭’ , if f −1(Y) in 𝕭’ is a fuzzy ideal of 𝕭’

**Proof.** For any *u* ∈ 𝕭’, there exists *𝛼* ∈ *𝕭’* such that *f* (*𝛼*) = *u*

Then *𝛗*(*u*) = *𝛗*(*f* (*𝛼*)) = *f* −1(*𝛗*)(*𝛼*) ≤ *f* −1(*𝛗*)(1) = *𝛗*(*f* (1)) = *𝛗*(1)

Therefore, *𝛗*(*u*) ≤ *𝛗*(1)

Let *u, v* ∈ 𝕭’. Then *f* (*𝛼*) = *u* and *f* (*𝛽*) = *v* for some *𝛼, 𝛽* ∈ *𝕭’*.

Thus, *𝛗*(*u*) = *𝛗*(*f* (*𝛼*)) = *f* −1(*𝛗*)(*𝛼*)

≥ *min*{*f* −1(*𝛗*)(*𝛼* ∗ *𝛽*)*, f* −1(*𝛗*)(*𝛽*)}

= *min*{*𝛗*(*f* (*𝛼* ∗ *𝛽*))*, 𝛗*(*f* (*𝛽*))}

= *min*{*𝛗*(*f* (*𝛼*) ∗ *f* (*𝛽*))*, 𝛗*(*f* (*𝛽*))}

= *min*{*𝛗*(*u* ∗ *v*)*, 𝛗*(*v*)}

Therefore, *𝛗*(*u*) ≥ *min*{*𝛗*(*u* ∗ *v*)*, 𝛗*(*v*)}

Then Yis a fuzzy ideal of 𝕭’

**Theorem 4.12.** Let X and Ybe fuzzy ideals of 𝕭, then X × Y is a fuzzy ideal of 𝕭 × 𝕭

**Proof**. For any (*𝛼, 𝛽*) ∈ *𝕭* × *𝕭*, we have

(*𝛗*1 × *𝛗*2)(1*,* 1) = *min*{*𝛗*1(1)*, 𝛗*2(1)}

≥ *min*{*𝛗*1(*𝛼*)*, 𝛗*2(*𝛽*)} for all *𝛼, 𝛽* ∈ *𝕭’*

= (*𝛗*1 × *𝛗*2)(*𝛼, 𝛽*)

Therefore, (*𝛗*1 × *𝛗*2)(1*,* 1) ≥ (*𝛗*1 × *𝛗*2)(*𝛼, 𝛽*)

Let (*𝛼*1*, 𝛽*1)*,* (*𝛼*2*, 𝛽*2) ∈ *𝕭’* × *𝕭’*. Then

(*𝛗*1 × *𝛗*2)(*𝛼*1*, 𝛽*1) = *min*{*𝛗*1(*𝛼*1)*, 𝛗*2(*𝛽*1)}

≥ *min*{*min*{*𝛗*1(*𝛼*1 ∗ *𝛼*2)*, 𝛗*1(*𝛼*2)}*, min*{*𝛗*2(*𝛽*1 ∗ *𝛽*2)*, 𝛗*2(*𝛽*2)}}

= *min*{*min*{*𝛗*1(*𝛼*1 ∗ *𝛼*2)*, 𝛗*2(*𝛽*1 ∗ *𝛽*2)}*, min*{*𝛗*1(*𝛼*2)*, 𝛗*2(*𝛽*2)}}

= *min*{(*𝛗*1 × *𝛗*2)((*𝛼*1 ∗ *𝛼*2)*,* (*𝛽*1 ∗ *𝛽*2))*,* (*𝛗*1 × *𝛗*2)(*𝛼*2*, 𝛽*2)}

Therefore, (*𝛗*1 × *𝛗*2)(*𝛼*1*, 𝛽*1) ≥ *min*{(*𝛗*1 × *𝛗*2)((*𝛼*1 ∗ *𝛼*2)*,* (*𝛽*1 ∗ *𝛽*2))*,* (*𝛗*1 × *𝛗*2)(*𝛼*2*, 𝛽*2)}

Hence, X× Yis a fuzzy ideal of *𝕭* × *𝕭*

**Theorem 4.13.** Let X and Y be the twofuzzy sets in 𝕭 such that X × Y is a fuzzy ideal of 𝕭 × 𝕭, then

1. Either 𝛗1(1) ≥ 𝛗1(𝛼) or 𝛗2(1) ≥ 𝛗2(𝛼) ∀ 𝛼 ∈ 𝕭’
2. If 𝛗1(1) ≥ 𝛗1(𝛼) ∀ 𝛼 ∈ 𝕭’, then either 𝛗2(1) ≥ 𝛗1(𝛼) or 𝛗2(1) ≥ 𝛗2(𝛼)
3. If 𝛗2(1) ≥ 𝛗2(𝛼) ∀ 𝛼 ∈ 𝕭’, then either 𝛗1(1) ≥ 𝛗1(𝛼) or 𝛗1(1) ≥ 𝛗2(𝛼)

**Proof.** (i) Assume that *𝛗*1(*𝛼*) *> 𝛗*1(1) and *𝛗*2(*𝛽*) *> 𝛗*2(1) for some *𝛼, 𝛽* ∈ *𝕭’*.

Then (*𝛗*1 × *𝛗*2)(*𝛼, 𝛽*) = *min*{*𝛗*1(*𝛼*)*, 𝛗*2(*𝛽*)}

*> min*{*𝛗*1(1)*, 𝛗*2(1)}

= (*𝛗*1 × *𝛗*2)(1*,* 1)

⟹ (*𝛗*1 × *𝛗*2)(*𝛼, 𝛽*) *>* (*𝛗*1 × *𝛗*2)(1*,* 1) ∀ *𝛼, 𝛽* ∈ *𝕭’,*

which is a contradiction*.*

Hence (i) is proved.

1. Again assume that *𝛗*2(1) *< 𝛗*1(*𝛼*) and *𝛗*2(1) *< 𝛗*2(*𝛽*) ∀ *𝛼, 𝛽* ∈ *𝕭’*

Then(*𝛗*1 × *𝛗*2)(1*,* 1) = *min*{*𝛗*1(1)*, 𝛗*2(1)}

= *𝛗*2(1)

Now,(*𝛗*1 × *𝛗*2)(*𝛼, 𝛽*) = *min*{*𝛗*1(*𝛼*)*, 𝛗*2(*𝛽*)}

*> 𝛗*2(1)

= (*𝛗*1 × *𝛗*2)(1*,* 1)*,* which is a contradiction*.*

Hence (ii) is proved

1. The proof is similar to (ii)

# **References**

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