Nonlinear Dynamics of Reaction Diffusion systems: Turing’s Analysis

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ABSTRACT

The aim of this chapter is to give a brief outline about the Nonlinear Dynamics of reaction diffusion systems. When diffusion is coupled with chemical kinetics some interesting spatio temporal structures arise such as stationary, spatially-varying concentration or patterns, spirals, traveling waves, targets etc. In view of Alan Turings phenomenological theory on Morphogenesis the pattern formation of reaction diffusion systems can be understood. The theoretical prediction of spatio-temporal instability and its experimental demonstration is illustrated in this chapter.

Keywords—stability, Nonlinear dynamics, diffusion, bifurcation, instability.

#  INTRODUCTION

 The study of Non linear dynamics is an important tool for understanding various bio physical and chemical phenomena such as glycolytic oscillations, Ca+2 oscillations, circadian cycles, cell cycles and so on[1-4]. In one of the previous book chapters the emergence of oscillation from Nonlinear dynamical point of view had already been discussed [5]. When diffusion is introduced into reaction kinetics, nonlinear dynamics of the system becomes significantly richer. One encounters stationary, spatially-varying concentration or patterns, spirals, traveling waves, targets, wave propagations etc. These spatial and spatio-temporal structures can be realized by coupling the reaction part with diffusion at discrete spatial points [6,7]. The dynamics of these systems are governed by reaction-diffusion equations. Alan Turing, in one of his seminal papers on the theory of Morphogenesis [8], had revealed the mystery behind the structural evolution of patterns in biological systems, such as, coat patterns of animals like tigers, zebras, pigmentation patterns of fish, spots, stripes and spiral patterns of various biological species. Turing’s analysis illustrates that the necessary and sufficient condition for generation of pattern is the disparity in diffusivities of the reacting species. Because of the lack of realization of an open thermodynamic system, experimental observation of Turing pattern remained illusive for nearly four decades after its theoretical prediction. With the development of suitable experimental techniques unambiguous experimental evidences on Turing pattern were clearly established in the last decade of twentieth century[9,10]. Thus the theoretical prediction of spatio-temporal instability and its experimental demonstration have now opened up a new horizon in the field of reaction-diffusion systems. In this present chapter we will focus on some of the primary features of the spatio temporal structures.

# TURING’S ANALYSIS

Let us think of two reactants A and B. In absence of diffusion they react to reach some steady state. Now this steady state can be stable or not. Turing raised the question whether diffusion can bring in instability to an otherwise stable steady state. Analysis revealed that this is possible provided the rates of diffusion of the two species widely differ. The idea is novel as it contradicts the well known stabilizing role of diffusion. Before going through the analytical details it would be nice to grasp the details of the theory intuitively. Consider an auto-catalytic reaction mechanism which involves an ’activator’ that diffuses slowly compared to ‘inhibitor’. Let by shear chance a small region of space sees a sudden rise in activator concentration. This in turn enhances the reaction rate and as a consequence concentration of both the species grow. Now the inhibitor diffuses out at a faster rate compared to the activator. The result is that a small region in space becomes richer in activator surrounded by an area richer in inhibitor. Thus any small perturbation of concentration grows in time and an inhomogeneity results. We now present the analysis of Turing in a somewhat modified form.

   

Figure 1: Different types of coat patterns generated in animals fish. This is predicted theoretically[10]

 If we consider a model of two chemical species in one dimension where one is acting like an activator *u*(*x,t*)
and other one is an inhibitor *v*(*x,t*) then the corresponding reaction-diffusion equations are given by

$$u\_{t}=γf\left(u,v\right)+∇^{2}u$$

 $v\_{t}=g\left(u,v\right)+d∇^{2}v$ (1).

where *d* = *Dv/Du* , is the ratio of the diffusion coefficients of the species *v* and *u*, respectively and *g* is the constant related to the length scale of the problem. The parameter space has to be chosen in such a way that in absence of diffusion both *u* and *v* tend to linearly stable steady state which is homogeneous in nature. In presence of diffusion where $D\_{v }\ne D\_{u}$the spatially stationary inhomogeneous patterns may develop under certain conditions by diffusion-driven instability. The uniqueness of this concept lies in the fact that diffusion
which is usually considered as a stabilizing process is responsible for causing the instability. We begin by looking for the necessary and sufficient conditions for diffusion-driven instability of the homogeneous steady state and the initiation of spatial pattern for such a general system given by Eq 1. Following Turing we impose the zero flux boundary conditions. The relevant homogeneous steady state ($u\_{0},v\_{0}$) of Eq 1 is given by

$$f\left(u\_{0},v\_{0}\right)=0$$

$$g\left(u\_{0},v\_{0}\right)=0$$

 (2).

Now in absence of diffusion *u* and *v* satisfy

$$u\_{t}=γf\left(u,v\right)$$

$$v\_{t}=g\left(u,v\right)$$

 (3).

Linearizing around the steady state $\left(u\_{0},v\_{0}\right)$ i.e. assuming $u=u\_{0}+δu$ and $v=v\_{0}+δv$, we
ultimately arrive at the dynamical equations for the perturbations $δu$ and $δv$as,



where *A* is known as the stability matrix comprising of partial derivatives of *f* and *g* with respect to *u* and *v* respectively evaluated at the steady state ($u\_{0},v\_{0}$) . *y* is defined as



We now look for solutions of the form $y\~e^{λt}$ where *λ* is the eigenvalue. For the steady state to be linearly stable the real part of the eigenvalue should be less than zero which assumes that the perturbation *y→*0 as t tend*s* to infinity. Proceeding as usual we are led to the following algebraic equation for the eigenvalues

$$λ^{2}- γ\left(f\_{u}+g\_{v}\right)λ+γ^{2}(f\_{u}g\_{v}-g\_{u}f\_{v})=0$$

From the above equation it is quite evident that the linear stability of the steady state is
guaranteed if $f\_{u}+g\_{v}$< 0 and $f\_{u}g\_{v}-g\_{u}f\_{v}>0. $Since the steady state of a system is determined by the kinetic parameters the above mentioned inequalities define the stability regions for the homogeneous steady state . We now return to the full reaction-diffusion system and linearize it about the steady state to obtain

 

Here we allow the small perturbations in *u* and *v* i.e. *du* and *dv* to grow spatially as well temporarily around steady state as *du∼*$ e^{λt}$cos*kx* and dv accordingly, where *k* is the wavenumber. Application of zero flux boundary condition then results in *k* = $\frac{nπ}{a}$. *n* is an integer and *a* determines the domain size in one dimension. Putting the aforesaid form of the perturbations in we ultimately arrive at an equation

 

Where .

From the above equation one comes out with a necessary condition for instability in presence of
diffusion, *Reλ >* 0

 $df\_{u}+ g\_{v}>0$

The above equation demands that *d ≠* 1 and $f\_{u} $ and $g\_{v}$must have opposite sign. Above equation is necessary but not the sufficient condition for *Reλ >* 0 since it requires that *h*(*k*2) must be negative for some non-zero *k* i.e. the minimum *h* must be negative. The condition forwhich *h*(*k*2) *<* 0 for some $k^{2}\ne 0$ is

 

It is also possible to extend the treatment for two and three dimensional reaction-diffusion systems and depending on the nature of nonlinearity one may observe various types of patterns ranging from stripes to spots for the systems obeying the conditions discussed above.

 **III. SOME REACTION DIFFUSION SYSTEMS EXHIBITING TURING’S PATTERN**

## **The pigmentation fish model**

It is a two variable reaction diffusion sytem. It was first proposed by BArio et al as an alternative approach to mechano-chemical models where pattern arises due to physical interaction between cells with external surrounding leading to cell aggregation and differentiation. The equations are given by



where *α*, *β*, *γ*, $r\_{1}$, $r\_{2}$ are the given parameters of the dynamics. *d* is the length scale. The choice of reaction terms is motivated by requirement of conservation of certain chemical species and nonlinearity which determines the specific unstable modes to dominate for the selection of a typical pattern when Turing instability sets in. Since in the absence of diffusion the system admits one more solution at *v* = *-*( *α* + *γ*)*u/*(1 + *β*) which follows simply from homogeneous steady state conditions on the above equations. the state (0,0) can be ensured as the only uniform steady state by setting the parameter *α* = *-γ*. The complex pattern generated with this model under various conditions bear striking resemblance with pigmentation patterns observed in a number of fish species. For details we refer to Bario *et al.*



Fif 2: Generation of patterns in Fish as a result of different activator inhibitor concentrations [9]

## **Gierer Meinherdt model**

To take into consideration the two central features of pattern forming phenomena, viz. the
local self-enhancement and the long range inhibition, Gierer and Meinhardt introduced the
two species (A and B are the concentrations of the two reacting species) model.



In this model the species B is an antagonist and consequently *DB >> DA* which is the condition for formation of spatial instabilities. The coefficients $μ\_{A}$and $μ\_{B}$are the removal rates. The basic production terms are given by $σ\_{A}$and $σ\_{B}$, the cross reaction coefficients are given by $ρ\_{A}$and $ρ\_{B}$. The constant *KA* is called the saturation constant and is believed not that necessary for pattern forming instability to develop, rather it determines the shape of
the pattern.

There are some experimental systems which have been studied to illustrate the instabilities but we will elaborate those in some other reviews rather we will focus on some different tyes of instabilities eg spirals.

**IV** **Spirals**

Spirals are fascinating spatio-temporal objects with broken circular symmetry in excitable media. In a typical two-dimensional medium a wave originating at a point may form concentric circles, since the wave travels with same velocity in all directions, producing target patterns. When the pattern gets locally disrupted the waves tend to curl forming typical spirals. Spirals have been ubiquitous in several areas in complex media, involving living
systems as well. This is because physical heterogeneity in these spatially extended excitable systems is so common that most often spirals are born out of targets. Spirals have been detected in aggregating slime molds , carbon monoxide oxidation on single crystals of platinum, developing frog eggs, heart muscle, ferrocin-catalyzed BZ reaction, Gierer-Meinhardt model and other related systems. However, the most studied of all these systems is Beluosov-Zhabotinskii reaction.

  

Fig 3: Theoretical prediction of spiral patterns generated in CDIMA system which has quite relevance with experimental observations[13].

Before going to review some of the mathematical techniques that are in frequent use it is
necessary, however, to make it clear what precisely we mean by the term spiral. In the case of BZ reaction, it is a rotating, time-periodic, spatial structure of reactant concentrations as noted by Murray[6,11]. If one stands at the center of a spiral he would see a periodic wave train is passing by him since every time the spiral turns a wave front moves past him. A simple rotating spiral is described by a periodic function of the phase *φ* with

 *φ* = ψ (t) + m (θ) ± ψ (r)

where ψis the frequency, *m* is the number of arms on the spiral and ψ (r) is a function which
describes the type of spiral. The *±* in the m (θ) term determines the sense of rotation. Numerous authors have investigated spiral wave trains of general reaction diffusion mode. These involved analyses usually make use of asymptotic methods. The *λ-w* system has been very extensively used as a model system because of the relative algebraic simplicity of the analysis. In this section we present some representative solutions for the *λ -w* system for illustration, following Murray, keeping in mind the direct relevance to real reaction-diffusion mechanism. The *λ -w* reaction diffusion mechanism for two reactants is



where *w*(*A*) and *λ* (*A*) are real functions of A.

If we consider w as a complex function of A and re do the analysis we will get the following dynamical equations as a solution of u and v.

 

Another way to deal with the problem of spiraling is to arrive at a linear amplitude equation from the solvability criterion applied at first order in a perturbation expansion of the model involved using multiple scales. It can then be shown that a general solution of a spiral exists for the amplitude equation in a region of phase space where a homogeneous oscillatory state is stable. Targets and stars are some special cases.

 **V CONCLUSIONS**

So far we have discussed the generation of different patterns in animal, fish etc. It is an analytical support to morphogenesis. Non linear dynamical studies have utmost strength to describe the pattern generations. Now to differentiate between spots, stripes and spirals we have to study the effect of different nodes on the reaction diffusion system. We can also study the effect of external noise, electric field, magnetic field , stochasticity on the systems. Worthmentioned will be to tune the pattern generations above any critical threshold through any of the external applications. In an upcoming issue it is planned to elaborate the effects on reaction diffusion systems.

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