**Dynamics of Interaction of One Prey and Two Competing Predators with Population Heterogeneity**

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**Abstract*:*** A mathematical model of two predators competing for a single prey system incorporating population heterogeneity has been proposed and analyzed by extending the system of coupled logistic equations for three species system. The local stability analysis of all the equilibria has been carried out. It is observed that equilibrium points are globally stable in Varies 2-D planes. Through analytical results, it is observed that the interior equilibrium of one prey and two competing predators in prey dependent case is always unstable. At the end, discussed the dynamical behavior of a three species interaction with the help of some numerical examples by using MATLAB to support the analytical presentations.

**Keywords:**Three species, Equilibria, Jacobian matrix.

**1.1 Introduction:**

In ecological-mathematical modelling, extremely the most imperative branch is of predator-prey type models. Among the important themes for mathematical and ecological modelling, the key theme is the relationship between prey and predator and it will remain continue to be a principal theme due to its universal existence and importance [41]. Interactions between predators and prey are often sophisticated behavioral games in which the decisions of each player are influenced by the decisions of the other. Both players play to maximize their own physical fitness or survival [1].

 Lotka-Volterra model is the initial predator prey system. Since then a number of improvements have been suggested to the primary model, including adding the predator saturation term, a prey which is self-competition term, a predator competition term and by taking distinct functional types etc.. [2]. These two species predator-prey models are the simplest form of mathematical models where there is interaction between a single prey and predator. While there is a principle in the ecosystem that models can be extended to multiple prey predator species systems [3].

A lot of research work has been done in the past in which authors studied extensively the extinction and co-existence of interacting species. The exploitation of a single resource by multiple competitors has received much attention and many authors have attempted to describe these situations mathematically [4-10]. Particularly, Hsu [6] initiated and examined a model containing a prey and two competing predators. The author has been observed in analysis that interference will affect the prey predator system. The authors Freedman and waltman [4] have considered food webs and studied in three levels, which are comprising of a single prey and two competing predators, and also two competing prey species affected by single predator. Cushing [11] has considered one prey population and two predator population system and initiated with a way that two predators which are competing feeding on one prey the resource of renewable and system is in the time of periodic. Dubey and Dass [12] investigated a Gauss- type mathematical model in which two predator populations competing with interference for a finite prey species. Dubey and Upadhyay [13] have also described a model in which one prey and two predator system considering ratio predator dependent rate growth.

 Reddy and Ramacharyulu [14] carried out the stability analysis of the system by considering two predator populations which are competing for one prey.

In the comparison of considering the influence of different type of conditions for predator-prey system a little research work has been done for the systems in which the population heterogeneity is being considered. Models incorporating heterogeneity are frequently used to describe the movement of population among patches. The interacting population is obviously very interesting thing to consider for the dynamics of the ecological system is influencing by the migration or dispersal of prey and predator [3].

Considering the above things, mainly in this chapter, an interacting prey predator model which is comprising of two predators and a single renewable resource prey in patchy habitat is proposed and analysed by extending the Kuno’s competitive equations for the three species ecosystem. Our projected mathematical system is carefully defined and the predators of the model in the condition by competition for exploiting the prey species. It’s noted that the outcomes of the system consisting of two predator populations and one prey population which is dependent function is closely known [6,12, 13]. In this particular chapter mainly to study the changes or dynamics of the qualitative ways of the system by incorporating the population heterogeneity.

**1.2 Background:**

Competition is an interaction of ecological systems which happens when two or more prey predator species depend on the same limited resource. Direct competition is exceptionally rare in nature; when two species compete, mostly one will dislocate the other, or they will adapt to utilize different resources and consequently avoid competition. Another ecological process related to competition is crowding, that is the habitat are densely populated than the normal. Therefore, the crowding effect is equivalent to the spatial heterogeneity or population heterogeneity, which emerges due to the number of processes inclusive of genetics, dispersal and resource availability. This process is exhibited to the effect of competitive procedure like an exclusion of various interacting prey predator species and coexistence. Intra-specific means the crowding effect denoted by as defined by Lloyd, which means that the average count of other animals per species of their own habitat per patch, and which is expressed as given below:

where denotes the count of animals at patch of species.

A patch is any discrete part of a population formed on either width or area. Patches explained on the basis of an area are just the sections across the geographical lines of a population’s habitat whereas the patches explained on the foundation of area are simply the number of individuals forming ecologically a meaningful unit. Patches based on size are ecologically more meaningful than the patches based on area as the area based patch in an animal zone containing no animal in it has ecologically no meaning. Under this approach, a patch may be defined as a small group of animals which do not interact with the animals of other groups except the animals of their own group in an ecologically meaningful way.

Iwao derived a simple linear model to express the connection between the average crowding and the average density of individuals per patch of habitat as

Although equation is represents a linear regression likewise it could be regarded while the linear functional connection. A distinctive property of Iwao’s model is, when population is allocated according to the Poisson, uniform or negative binomial distribution its parameters (intercept) and (slope) have been shown to acquire particular values

**Table 1:**

The values of & for the population of different spatial distributions

Distribution

Uniform distribution

 Poisson (Random) distribution

 Negative binomial distribution

 Contagious distribution

Regular (underdispersed) distribution

Besides intra-specific mean crowding Lloyd also explained an inter-specific mean crowding denoted by means the average count of animals of habitat per animal of other animals and is calculated by the formulae

where and are the number of animals of the and species in the patch of habitat. This index is required whenever we have a need to extend the model dealing with the interaction of single species to multiple species.

The logistic equation is mostly used for the betterment of an exponentially expressing growth the species. However, it is also well recognized that the growth of logistic equation acts impractical in different circumstances. The expression of growth of logistic equation of population individuals usually appears in the form

here represents the first derivative of species number or area with the time, represents the intrinsic growth or generative rate of the given population and K represents the environment carrying capacity. Carrying capacity is referred as the maximum size of population that the environment can support. In many situations this form of the above logistic expression yields biologically absurd outcomes. The most remarkable limitation of is that even if the generative rate is fatalistic, that means which occurs in the situation when the early value is more than (Levin’s paradox) the population still increases indefinitely. Another model of the constraints of expression is presumes that the particulars will be uniformly distributed in the given environment. The possibility will overcome at first limitation, Levin’s paradox, for utilize the Verhulst’s logistic expression,

where denotes the positive constant which is expressing the intrinsic limitations of population or species growth . Kunoused logistic equation which is the foundation of mathematical design to solve the absurd presumption of the species which are distributed homogeneously.

To extend the logistic model given in equation for simulating the competing result of one species upon another, Water’s et al., inused the structure of coupled logistic which are in growth expressions. The given system explains the population or species growth for competing species:

where and are two respective species and parameters represents the competitive effect of the species on the .

Water’s et al. then derived the predator-prey case for the model by reversing the sign of as

Where, and represents the species or population density of prey and predators respectively .

**1.3 Extension to three dimensions:**

To generalize the model to three dimensions it is necessary to model the competitive interactions of particular species by adding the term, that shows the effect of populations on the intrinsic reproductive rate of the species. Thus the resulting model describes the growth of population for three competing population is

Now we obtain a predator-prey example of by considering population heterogeneity in the tri trophic ecosystem consisting of both predators competing for the single prey to investigate whether it will facilitate the existence of stable equilibrium in this case or not.

**1.4 Extension of predator-prey example:**

To describe prey - predator system (9), represents the amount of prey common to two predators and which are competing with each other and by reversing the sign’s of and so that

If the predators are taken as a function of prey only, subscripts of & will be needless because these parameters apply to prey only. Therefore by removing the and for predators & rewriting the logistic growth only to prey species, but keeping predator species to increase entirely like an expression of prey species yields

 (11)

If we consider the death rates for the predators and as and respectively then the resulting equations become

 (12)

In (12) assumed that prey & predators both are homogeneously distributed. So the predator prey system given in could be extended to a patchy species like

where are the density of prey & two species of predators at of n patches. While dividing equation by yields

 (14)

In above equation is the inter-certain mean packed of the species upon the ,, and , where represents the intra specific crowding effect for prey species, showing heterogeneity effect upon its rate of reproduction. By (14) of & , we get

S

 (15)

Using the relationship and by multiplying the above equation on both sides by and , we obtain

 (16)

 .

When we get

 (17)

 .

With the conditions of initial are:

 .

**1.5 Boundedness of the Model:**

Lemma: The solutions (17) institute the are bounded.

Proof. Let

The time derivative along a solution (17) is

For

Now, if we choose:

then the right side of equation (19) is bounded for all ( ). Thus we define a with .

By applying the differential inequality theorem, we get:

and for

So we have:

**1.6 Equilibria Analysis:**

It could be checked that the model has five positive equilibria, which are,

given by . The point clearly exists. Now will discuss and equilibria as follows:

**(i) For:**

Here is a positive value from the following expression

Solving we get

 This equilibrium state would exist only if .

**(ii) Existence of:**

Here are having non negative solutions by these two algebraic expressions:

Solving above two algebraic equations, we get

 This point exists only if .

**(iii) Existence of:**

Here are non-negative results from these algebraic expressions

Solving above two algebraic equations, we get

This equilibrium state would exist only if .

**(iv) Existence of:**

Here are non-negative solutions by these algebraic equations

 Solving above three algebraic equations, we get

 .

This equilibrium state would exist only if

 ,

 and .

**1.7 Stability Analysis:**

The Jacobian of system (17) denoted by A is constructed as

Let

then,

Therefore,

 (29)

Here, = O11

The characteristic equation is

The equilibrium point will be stable only when the results of (30) are negative if they are real or have negative real parts if they are complex.

**(i) Stability analysis of fully washed out equilibrium point ():**

The Jacobian matrix (29) evaluated at is

The characteristic equation corresponding to is

Therefore, the three eigen-values are:

 and.

Clearly, and

For stability should also be less than zero

 if

Thus the equilibrium state is stable only if

Or, , otherwise unstable.

**(ii) Stability analysis of prey only surviving equilibrium state :**

The Jacobian matrix (29) processed at is

The characteristic equation corresponding to is

 (33)

Therefore, the eigen-values are

 and.

For stability it is necessary that all the three eigen-values must be negative. For it is necessary that

1.

or

1.

or

1.

or

If all these three results hold, then the point will be a stable node, or else it will be a saddle point.

**(iii) Stability analysis of predator (washed out equilibrium state (:**

The Jacobian matrix (29) evaluated at is

Where O22=

The characteristic equation corresponding to  is

One eigen-value of is , & the rest of two eigen values can be get from the following expression:

 (38)

whose sum of roots is negative & product is positive.

Thus, the roots of (35) are real and negative or complex conjugates having negative real parts. So, the equilibrium point is asymptotically stable only if

or, otherwise unstable.

**(iv) Stability analysis of predator () washed out equilibrium state :**

(29) processed at is

O33=

The characteristic polynomial corresponding to is

 One eigen-value is and other both values can be evaluated from the following expression:

(41)

whose sum of roots is is negative and its product of roots is positive.

Thus, the roots of equation (38) are real & negative or complex conjugates having negative real parts. So, the equilibrium state is asymptotically stable only if

or otherwise unstable.

**(v) Stability analysis of co-existing state (:**

The Jacobian matrix (29) evaluated at is

The characteristic equation corresponding to is

+

or, (43)

where

 and

According to Routh-Herwitz’s criteria, is locally asymptotically stable when it holds the given rules

 (44)

 and

Since, so it does not satisfy all the above three conditions of equation (44). Hence we can conclude that is unstable saddle point.

Therefore, it is concluded that the point (E4) is always unstable as mentioned in the theorem 3.4 of Hsu [6].

**1.7 The Global stability of equilibria:**

Here, we prove the global stability by using 2-D planes.

**Theorem 1:** The point is globally asymptotically stable in plane.

Proof. Assume

It is obvious that if and .

Now, we denote

Then,

 = .

Thus, for all . Hence by the Bendixson-Dulac rule, there will be no periodic orbit in the non-negative quadrant of plane.

This completes the proof.

**Theorem 2:** is globally asymptotically stable in plane.

Proof. Let

It is obvious that if and .

Now, we denote

Then,

 = .

Thus, for all . Hence by the criterion of Bendixson-Dulac, there will be no periodic orbit in the non-negative quadrant of plane.

This completes the proof.

**Remark:** For the similar manner, observed at is stable which is globally asymptotically in and planes.

**1.8 The Numerical Simulation of the Model:**

In numerical section, we discuss the dynamical behavior of a tri species interaction comprising of one prey and two competing predators incorporating population heterogeneity with the help of some numerical examples by using the MATLAB.

**1.8.1. Uniformly distributed case:** In this case the values of are

 and

We select the following certain values for

, , , ; , ; , , ;

It is observed that the point is stable which is of locally asymptotically (see Fig. 1(a) and Fig. 1(b)).



Fig.1(a): Time graph around showing the stability behavior.



Fig.1(b): Phase graph around the equilibrium point showing the stability role.

Now we select the following values for

 , , , ; , , ; , , ;

It is observed that is locally asymptotically stable (see Fig. 2(a) and Fig. 2(b)).



Fig.2(a): Time graph around the equilibrium point showing the stability behavior.



Fig.2(b): Phase graph around the equilibrium point .

Now we select the parameter values for

 , , , ; , , ; , , ;

It is found that under the above set of parameters the equilibrium is locally asymptotically stable (see Fig. 3(a) and Fig. 3(b)).



Fig.3(a): Time graph around the equilibrium point showing the stability behavior.



Fig.3(b): Phase graph around the equilibrium point showing the stability behavior.

**1.8.2. Poisson (Randomly) distributed case:** In this case the values of are

 and

We select these values for

, , , ; , , ; , , ;

It is observed that is locally asymptotically stable (see Fig. 4(a) and Fig. 4(b)).



Fig.4(a): Time graph for explaining the stability.



Fig.4(b): Phase graph around the equilibrium point .

Now we select some values for

 , , , ; , , ; , , ;

It is seen that is locally asymptotically stable (see Fig. 5(a) and Fig. 5(b)).



Fig.5(a): Time series graph around the equilibrium point showing the stability behavior.



Fig.5(b): Phase graph around the equilibrium point .

Finally select some values for

, , , ; , , ; , , ;

It is seen that is locally asymptotically stable (see Fig. 6(a) and Fig. 6(b)).



Fig.6(a): Time series graph around the equilibrium point showing the stability behavior.



Fig.6(b): Phase graph for .

**1.9 Conclusion:**

A mathematical model of prey predator, that is two predators competing for a single prey system considering population heterogeneity has been proposed and analysed by extending the system of coupled logistic equations for three species system. The stability examination of all the equilibria has been discussed at locally. It is seen that the equilibrium points are globally stable in varies 2-D planes. From the stability of, only prey will survive and predators will tend to vanish under the conditions (34), (35) and (36). For, it is seen that only the preyand predator will survive and the population of the predator may die out under the condition (39). However from the point, only the preyand predator will survive and the population of the predator may die out under the condition (42). From theorem 3.4 of Hsu [6] it is identified that the species in prey dependent case is every time unstable. Indeed, from , it is observed is an unstable saddle point with two dimensional stable manifolds. At the end, discussed the emphatic behavior of all the species interaction with the help of some numerical examples by using MATLAB to carry the analytical presentations.

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