**Solution of Singular Perturbation Problems using Fourth Order Adaptive Cubic Spline**

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**ABSTRACT**

In this paper, using adaptive cubic spline, we have suggested a numerical scheme for solving a convection-diffusion problem having layer structure. The numerical scheme is derived with this spline and non-standard finite differences of the first derivative. The tridiagonal solver is used to solve the system of the numerical method. The analysis of convergence of the method is briefly discussed and the fourth order is shown. The numerical results of the examples were tabulated and compared to the existing computational results in order to support the higher accuracy of the proposed numerical scheme.

**Keywords:** singular perturbation; adaptive cubic spline; convection-diffusion problems; convergence.

**I. INTRODUCTION**

It is well known that many physical problems with many small parameters often involve the solution of boundary value problems. This paper deals with convection-diffusion boundary value problems involving small parameter. These problems are characterized by the inclusion of a small perturbation parameters𝜀 which multiply the second order derivative. In many fields of engineering and science, such types of problems exist such as chemical reactor theory, transport phenomena in chemistry, lubrication theory and biology.

A broad verity of books has been found in the literature for the convection – diffusion problems or singular perturbation problems (SPPs) [2,3,4,10,14]. One can refer a book on splines by Micula [9]. The survey papers [6, 8] provides a detailed research work on SPP problems. In [1], the authors suggested a difference schemes of second and fourth order based on cubic spline in compression for SPP. A variable-mesh second-order difference scheme via cubic splines is proposed to solve SPP in [5]. In the paper [7], authors usedthe artificial viscosity in B-spline collocation method to capture the layer behaviour of the problem. Phaneendra and Lalu [12] derived numerical scheme using Gaussian quadrature for the solution of SPP with one end layer, dual layer and internal layer. The authors in [13] extended the Numerov scheme to the SPP with first order derivative. Soujanya et al. [15] introduced a scheme having a fitting factor in Dahlquist scheme to get the solution of SPP having dual layers. Uniform difference schemes based on a class of splines are proposed by Stojanovic [16] for the solution of non-self-adjoint SPP.

In this paper, we present a fourth order finite difference method using adaptive cubic spline to solve singularly perturbed boundary value problems. We introduce a new parameter 𝜂 in the difference scheme to achieve fourth order convergence for the proposed problem. The paper is organized as follows: In section 2, Description of the problem along with conditions for layer behavior is given. In section 3, we define the adaptive spline function. In section 4, we describe the numerical method for solving singularly perturbed boundary value problems, in Section 5, the truncation error and classification of various orders of the proposed method are given. In section 6, we discuss the convergence analysis of the method. Finally, numerical results and comparison with other methods are given in section 7.

**II. DESCRIPTION OF THE METHOD**

Consider the convection-diffusion boundary value problem of the form

(1)

with boundary conditions (2)

Here is a perturbation parameter. The functions are assumed to be appropriately smooth in and , are finite constants. The layer exists in the vicinity of if over the domain , where *L* is positive constant. The layer exists in the vicinity of if over the domain

where is negative constant.

**III. ADAPTIVE CUBIC SPLINE**

With grid points in , consider the mesh such that , where for . A function interpolate at the grid points which depends on a variable , leads to cubic spline in and is named as adaptive spline function as . Following [13], If is an adaptive spline function then

(3)

where , and .

Solving Eq. (3) and using the interpolatory conditions , , we have

(4)   
where

where

The spline function on is acquired with replacing *i* by  in Eq. (4) and utilizing the first () or second derivative continuity condition of at , we get the following relationship:

(5)

Further the relations are given below for the adaptive spline

where

We also obtain (6)

**Remark:** In the limiting case when , we have

and the spline function (3) reduces to ordinary cubic spline.

I**V. NUMERICAL SCHEME**

At the mesh point , the suggested approach can be discretized by the convection-diffusion equation Eq. (1) as

(7)

The above equations shall be replaced in Eq. (6) and using the following approximations for the first order derivative of *s* at the mesh points

*,*

we get the following system

(8)

where

The tridiagonal system Eq. (2.8) is solved for *i =* 1, 2, …, *N*-1 to obtain the approximations of the solution *y(s)* at .

**V. TRUNCATION ERROR**

The scheme’s local truncation error in Eq. (2.8) as follows

Thus for different values of in the scheme (8), indicates different orders:

**Remarks:**

1. If , for any choice of arbitrarywith and for any value of , method is obtained for second order.
2. For,, fourth order method is derived.

**VI. ANALYSIS OF CONVERGENCE**

The convergence analysis of the suggested method to Eq. (1) is now being considered. The system of equations in the matrix form with the boundary conditions is

(9)

where and

where

and

where

for ,

are associated vectors of Eq. (9)

Let which satisfies the equation

(10)

Letbe the discretization error so that

.

By deducting Eq. (9) from Eq. (10), the error equation is developed as

(11)

Let and where are positive constants. If be the element of *F*, then

Thus for sufficiently small , we have

(12)   
 (13)

Hence is irreducible [15].

Let be the sum of the entries of the row of matrix, then we obtain

Let ,.

We have it has been confimed that for sufficiently small is monotone [14,15].

Hence

Thus from Eq. (11) we get

(14)

Let and we define

(15)

Since .

Hence (16)

(17)

Furthermore

(18)

By the help of Eqs. (15) - (18) and using Eq. (14) we get

(19)

Hence the method given in Eq. (8) is fourth order convergent for

,

**VII. NUMERICAL EXPERIMENTS**

Two point singular perturbation problems are investigated based on adaptive splineto establish the vitality of our proposed method computationally. These illustrations were considered because they were extensively explored in the research and had accurate solutions that could be compared. Maximum absolute errors in the solution are tabulated and compared with the existing method results which have demonstrated improvement.

**Example 1.**    
 where

**Example 2.**

is the exact solution

**Example 3.** with

The exact solution is

.

**Example 4**. with

This problem exhibits dual layers at

**VIII. DISCUSSIONS AND CONCLUSION**

In this chapter, we suggested a numerical scheme to solve a convection-diffusion problems using adaptive cubic spline. We introduce a new parameter in the difference scheme to achieve fourth order convergence for the suggested problem. We have obtained a three-term relation with the help of difference scheme which involves a parameter. Using this, we have solved the tridiagonal scheme obtained by the method using discrete invariant imbedding. The convergence of the method has been discussed for the standard test examples chosen from the literature. MAEs are provided to illustrate the effectiveness of the method and also presented point wise errors in the solution of the examples for the comparison shown in tables 1-.4 to support the method. The proposed fourth order algorithm has been found to produce better results.

**Table 1.1:** The MAEs in Example 1 for and

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Proposed Method

3.35(-4) 8.36(-5) 2.09(-5) 5.22(-6) 1.31(-6)

7.82(-4) 1.94(-4) 4.84(-5) 1.21(-5) 3.03(-6)

1.76(-3) 4.24(-4) 1.05(-4) 2.63(-5) 6.56(-6)

4.53(-3) 9.43(-4) 2.23(-4) 5.51(-5) 1.37(-5)

2.39(-2) 2.83(-3) 5.09(-4) 1.16(-4) 2.82(-5)

1.36(-1) 1.95(-2) 1.96(-3) 2.86(-4) 6.01(-5)

Results in [11]

8.12(*-*4) 2.03(*-*4) 5.07(*-*5) 1.26(*-*5) 3.17(*-*6)

3.53(*-*3) 8.79(*-*4) 2.19(*-*4) 5.48(*-*5) 1.37(*-*5)

1.50(*-*2) 3.68(*-*3) 9.17(*-*4) 2.29(*-*4) 5.72(*-*5)

6.75(*-*2) 1.54(*-*2) 3.77(*-*3) 9.37(*-*4) 2.34(*-*4)

2.66(*-*1) 6.83(*-*2) 1.55(*-*2) 3.81(*-*3) 9.48(*-*4)

6.92(*-*1) 2.68(*-*1) 6.87(*-*2) 1.56(-2) 3.83(*-*3)

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**Table 1.2.** The MAEs in Example 2 for and

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Proposed Method

1.24(-07) 7.76(-09) 4.85(-10) 3.05(-11) 2.60(-12)

2.00(-06) 1.25(-07) 7.80(-09) 4.87(-10) 3.06(-11)

3.24(-05) 2.00(-06) 1.25(-07) 7.80(-09) 4.87(-10)

5.42(-04) 3.24(-05) 2.00(-06) 1.25(-07) 7.80(-09)

0.0075 5.42(-04) 3.24(-05) 2.00(-06) 1.25(-07)

0.0586 0.0075 5.42(-04) 3.24(-05) 2.00(-06)

0.2255 0.0586 0.0075 5.42 (-04) 3.24(-05)

Results in [11]

4.77(*-*4) 1.19(*-*4) 2.98(*-*5) 7.45(*-*6) 1.86(*-*6)

1.92(*-*3) 4.79(*-*4) 1.19(*-*4) 2.99(*-*5) 7.48(*-*6)

7.87(*-*3) 1.92(*-*3) 4.79(*-*4) 1.19(*-*4) 2.99(*-*5)

3.45(*-*2) 7.87(*-*3) 1.92(*-*3) 4.79(*-*4) 1.19(*-*4)

6.75(*-*2) 1.54(*-*2) 3.77(*-*3) 9.37(*-*4) 2.34(*-*4)

3.51(*-*1) 1.35(*-*1) 3.45(*-*2) 7.87(*-*3) 1.92(*-*3)

6.00(*-*1) 3.51(*-*1) 1.35(*-*1) 3.45(*-*2) 7.87(*-*3)

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**Table 1.3**. The MAEs in Example 3

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Results in [12]

5.9701(-3) 3.3654(-3) 1.7391(-3) 8.7449(-4) 4.3697(-4) 2.1822(-4)

5.3525(-3) 3.2322(-3) 1.7219(-3) 8.7336(-4) 4.3719(-4) 2.1834(-4)

1.1177(-2) 2.9851(-3) 1.6827(-3) 8.6953(-4) 4.3725(-4) 2.1848(-4)

2.5867(-2) 2.6763(-3) 1.6161(-3) 8.6093(-4) 4.3668(-4) 2.1860(-4)

4.7842(-2) 5.5886(-3) 1.4925(-3) 8.4134(-4) 4.3477(-4) 2.1862(-4)

7.5829(-2) 1.2934(-2) 1.3381(-3) 8.0805(-4) 4.3046(-4) 2.1834(-4)

Proposed method

8.6799(-6) 5.4123(-7) 3.3837(-8) 2.1151(-9) 1.3218(-10) 8.1866(-12)

2.4578(-5) 1.5331(-6) 9.5748(-8) 5.9831(-9) 3.7392(-10) 2.3330(-11)

6.7327(-5) 4.3399(-6) 2.7061(-7) 1.6919(-8) 1.0575(-09) 6.6083(-11)

1.9559(-4) 1.2289(-5) 7.6654(-7) 4.7874(-8) 2.9915(-09) 1.8695(-10)

5.6901(-4) 3.3663(-5) 2.1700(-6) 1.3531(-7) 8.4593(-09) 5.2877(-10)

1.5105(-3) 9.7794(-5) 6.1444(-6) 3.8327(-7) 2.3937(-08) 1.4958(-09)

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**Table 1**.4 MAEs in Example 4

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Proposed method

6.6213(-03) 4.9949(-04) 2.9956(-05) 1.8527(-06) 1.1549(-07) 7.2202(-09)

5.3873(-02) 7.0956(-03) 5.2237(-04) 3.1287(-05) 1.9343(-06) 1.2056(-07)

2.1476(-01) 5.6298(-02) 7.3146(-03) 5.3245(-04) 3.1875(-05) 1.9704(-06)

4.6087(-01) 2.2024(-01) 5.7467(-02) 7.4197(-03) 5.3714(-04) 3.2149(-05)

6.7876(-01) 4.6685(-01) 2.2288(-01) 5.8040(-02) 7.4712(-03) 5.3941(-04)

8.2386(-01) 6.8315(-01) 4.6973(-01) 2.2419(-01) 5.8325(-02) 7.4966(-03)

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**Figure 1:** Solution Profile for with **Figure 2:** Solution Profile for with

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**Figure 3:** Solution Profile for with  **Figure 4:** Solution Profile for with

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