**Exploring Fuzziness of the Parameters in Predator-Prey Model with Harvesting: An Application of Hukuhara Derivatives**

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**ABSTRACT**

In today's ecological research, the uncertainty surrounding parameter values in mathematical models holds a crucial role. This research article introduces the application of the Hukuhara derivative technique for formulating predator-prey interactions with harvesting. We construct a two-dimensional predator-prey model that accounts for the uncertain behavior of both prey and predator populations. We formulate Fuzzy Differential Equations (FDE) for various scenarios: when both populations, prey (x(t)) and predator (y(t)), are i-gH differentiable; when x(t) is i-gH differentiable while y(t) is ii-gH differentiable; when x(t) is ii-gH differentiable and y(t) is i-gH differentiable; and finally, when both populations are in ii-gH differentiable states. We provide analytical descriptions of stability criteria for equilibrium points.Our findings highlight that the dynamic behavior of predator-prey interactions with harvesting hinges on parameter fuzziness. We conclude from this study that the mathematical model's dynamics are strongly influenced by the imprecise nature of these parameters, mirroring a more realistic representation of ecological systems. This research opens a novel dimension in ecological studies.

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Keywords: Predator; Prey; Harvesting; Fuzzy; Hukuhara derivative.

1. **INTRODUCTION:**

In the field of prey-predator system, Harvesting has immense importance as a biological phenomenon to study the management of renewable resources. Fishery and Forestry uses harvesting of prey population [1, 2, 3]. In the recent years, researcher are more interested to study the dynamical behavior of predator prey system with Harvesting in the management of renewable resources [4-10]. Fisheries and floristries has been exploiting renewable resources since the past few decades and has became a major concern in bio economic analysis [1, 11]. The dynamics of multy species harvesting models has demanded the time of many researchers [12, 13, 14, 15, 16]. Non-selective harvesting model is studied in [17, 18], though there is a problem on non-selective harvesting policy with logistic growth [1]. This type of model has maximum sustainable yield (MSY). The population will extinct if the harvesting of a population exceeds its MSY, and the decreased harvested species can be recovered if the harvesting rate is smaller than its MSY [19].

Harvesting Policy are of two types – one is constant quota harvesting and the other one is constant-effort harvesting. A constant number of individual in the population are harvested per unit time and in the constant-effort harvesting, fixed effort is applied per unit of time to catch the animals in a population [20]. Although a particular rule is not followed by harvesting. For example, more number of fishes are harvested in warmer seasons than in cold seasons [21].

Fuzzy differential equations (FDEs) have gained widespread attention across various fields such as engineering, economics, biology, and physics. In real-life scenarios, fuzzy set theory plays a crucial role in modeling diverse problems. Individual behaviors within a system often exhibit inherent variations, which are confined to a relatively small set representing the collective characteristic behavior of the group. To effectively quantify the subjective qualities under study, it becomes essential to assign values or degrees that satisfactorily represent these qualities. This task cannot always be accomplished solely through objective measurements or statistics. Consequently, when the state variables of a given demographic system exhibit inherent uncertainty, it becomes imperative to incorporate fuzziness into the system, representing this uncertainty through fuzzy variables.In brief, when constructing models to depict real-world situations, it is common to encounter imprecise or uncertain data. Relying on precise real-life data in such instances may lead to errors. Therefore, it is prudent to consider parameters as imprecise in these situations. Naturally, a question arises: does the model's behavior remain consistent with that of the crisp (non-fuzzy) model? Clearly, the behavior differs, necessitating an investigation of this behavior within this context.FDEs emerge as a potent tool for characterizing the dynamics and characteristics of harvesting models in predator-prey systems.

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Fuzzy logic's FL-generalization stands as a remarkable contribution, bridging the gap between crisp sets and those reliant on fuzzy sets. One prominent manifestation of this generalization is Fuzzy Differential Equations (FDEs), a cornerstone of fuzzy mathematics. When all coefficients or parameters within the boundary conditions of a differential equation adopt a fuzzy nature, it qualifies as a fuzzy differential equation [22]. The credit for proposing fuzzy set theory goes to Zadeh, as it encompasses the concept of membership functions and addresses uncertainty [23]. In recent years, fuzzy set theory has surged in popularity within mathematics due to its diverse applications across various scientific domains. Fuzzy differential equations have proven invaluable for modeling numerous dynamic systems, accommodating the inherent uncertainty associated with parameters [24]. A comprehensive exploration of prey-predator system harvesting, incorporating fuzziness, can be found in [25].

The concept of fuzziness holds a pivotal role in dynamical systems, with the Hukuhara derivative emerging as a pioneering concept for addressing fuzzy differential equations [26]. Fuzzy differential equations were first established in 1978 [22], sparking extensive research efforts to develop and apply this concept [22]. Notably, the solutions of crisp models and fuzzy differential equations exhibit significant disparities, motivating numerous researchers to generalize their findings [24, 27, 28–40].First-order fuzzy differential equations pose particular challenges, as solutions vary depending on the interpretation [41–43]. Several methods, including generalized differentiability concepts and operator methods, have been devised for solving both first and second-order fuzzy differential equations [44–48]. To tackle fuzzy differential equations numerically, the generalized Euler approximation method finds utility [49,50].

In this research article, our primary objective is to model a predator-prey system with harvesting using fuzzy differential equations and to analyze the model under the more realistic assumption that both populations are uncertain in nature. In Section 2, we provide an overview of the fundamentals of fuzzy differential equations. Next, in Section 3, we introduce the mathematical model for predator-prey interaction with harvesting.Section 4 is dedicated to fuzzy differential equations, considering different scenarios where both populations, namely the prey population (x(t)) and predator population (y(t)), exhibit varying degrees of differentiability. These scenarios encompass cases where x(t) is i-gH and y(t) is ii-gH differentiable, as well as scenarios where x(t) is ii-gH and y(t) is i-gH differentiable. We also present the model when both populations are in ii-gH differentiable states and provide the corresponding results.Finally, we conclude the paper with a discussion and draw our overall conclusions.

**2. PRELIMINARIES:**

**2.1. Triangular fuzzy number [51]:**

 The membership function of a triangular fuzzy number M, denoted as (m\_1, m\_2, m\_3), is defined as follows:

n(x) = $\frac{x-m\_{1}}{m\_{2}-m\_{1}}$ , $m\_{1}\leq x\leq m\_{2}$

 = $\frac{m\_{3}-x}{m\_{3}-m\_{2}}$ , $m\_{2}\leq x\leq m\_{3}$

 = 0, $m\_{1}\geq x, m\_{3}\leq x$

**2.2.α-cut of triangular fuzzy number [51]:**

The α-cut of triangular fuzzy number M = ($m\_{1}, m\_{2},m\_{3}$), $∀$α$\in $ [0,1] is given by

 $M\_{α} = [m\_{1}+α(m\_{2}-m\_{1}), m\_{3}-α(m\_{3}-m\_{2})].$

**2.3. Generalized Hukuhara difference [52]:**

Let us define the generalized difference of A and B as the set C $\in M(X)$,where A and B belong to the set $M(X)$, and this is characterized by the following condition:

A$Θ\_{g}$B = C $⟺\left\{\left(i\right)A=B+C\right.$

 $\left\{or (ii)B=A+(-1)C\right.$

**2.4. Generalized Hukuhara derivative for fuzzy valued function [53]:**

The function g:(a,b) $\rightarrow R\_{F}$ is said to be strongly generalized Hukuhara differentiable or gH-differentiable at the point p0where p0$\in $ (a,b), if g’(p0) $\in R\_{F},$ such that

1. $∀ϵ$> 0 sufficiently small, $∃ $g(p0+$ϵ$)$Θ\_{g}$g(p0),g(p0)$Θ\_{g}$ g(p0-$ϵ)$ and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g(p\_{0}+ ϵ)Θ\_{g}g(p\_{0})}{ϵ}$ = $\lim\_{ϵ\to 0}\frac{g(p\_{0}) Θ\_{g} g(p\_{0}- ϵ)}{ϵ}$

or (ii) $∀ϵ$> 0 sufficiently small, $∃$g(p0)$Θ\_{g}$g(p0+$ϵ$), g(p0-$ϵ$) $Θ\_{g}$ g(p0$)$ and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}\right)Θ\_{g} g(p\_{0}+ ϵ)}{-ϵ}$ = $\lim\_{ϵ\to 0}\frac{g(p\_{0}- ϵ) Θ\_{g} g(p\_{0})}{-ϵ}$

or (iii) $∀ϵ$> 0 sufficiently small, $∃$g(p0+$ϵ$)$Θ\_{g}$g(p0), f(p0-$ϵ$) $Θ\_{g}$ f(p0$)$ and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}+ ϵ\right)Θ\_{g}g(p\_{0})}{ϵ}$ = $\lim\_{ϵ\to 0}\frac{g(p\_{0}- ϵ) Θ\_{g} f(p\_{0})}{-ϵ}$

or (iv) $∀ϵ$> 0 sufficiently small, $∃$g(p0)$Θ\_{g}$g(p0+$ϵ$), g(p0) $Θ\_{g}$ g(p0-$ϵ)$and the limits

g’(p0) = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}\right)Θ\_{g}g\left(p\_{0}+ϵ\right)}{-ϵ}$ = $\lim\_{ϵ\to 0}\frac{g\left(p\_{0}\right)Θ\_{g}g\left(p\_{0}-ϵ\right)}{ϵ}$

**2.5.Strong and weak solution of fuzzy differential equation:**

Let us consider the fuzzy differential equation $\frac{d g(t)}{dt} = g\_{1}(t,x(t))$with the initial condition g(t0)=x0 which is a fuzzy number and let,$\tilde{g}(t)$ be the solution of this fuzzy differential equation. The parametric form of α-cut ofthe solution $\tilde{g(t)}$ is denoted by [$g\_{1}(t,α),g\_{2}(t,α)]$.

 The solution $\tilde{g(t)}$is termed as strong fuzzy solution if $g\_{1}(t,α)$ is increasing function,$g\_{2}(t,α) $is decreasing function and $g\_{1}(t,α)\leq g\_{2}(t,α),∀$α$\in $ [0,1]. Otherwise the solution is refered to as weak solution.

 Weak fuzzy solution is also a solution of the fuzzy differential equation, but our aim is to transform the weak fuzzy solution to strong fuzzy solution with the help of the formula

 $\tilde{g}(t,α) = \left[min \left\{g\_{1}(t,α),g\_{2}(t,α)\right\}, max\left\{g\_{1}(t,α),g\_{2}(t,α)\right\}\right]$.

1. **FORMULATION OF THE MATHEMATICAL MODEL:**

This section is dedicated for mathematical model of predator-prey system with harvesting. Where x(t) represent the prey population and y(t) represent the predator population.

$$\frac{dx(t)}{dt}=rx(t)\left(1-\frac{x(t)}{k\_{1}}\right)-ax(t)y(t)-ex(t)$$

$\frac{dy\left(t\right)}{dt}=kax\left(t\right)y\left(t\right)-d\_{2}y\left(t\right)$(1)

$$With x\left(0\right)=x0 and y\left(0\right)=y0.$$

The parameters for the aforementioned system are as follows:

The intrinsic growth rate of the prey population, denoted as "r". The carrying capacity of the prey population, represented as "k1". The predation rate of the prey population by the predator, indicated by "a". The conversion rate of prey population into predator, denoted as "k". The harvesting coefficients for the prey population, given as "e". The death rate of the predator population, represented as "d2".

1. **PREDATOR-PREY MODEL WITH HARVESTING IN FUZZY ENVIRONMENT:**

Let's examine the fuzzy solution of the system of equation (1), denoted as (x̃(t),ỹ(t)). The α-cut of x̃(t) is defined as [x\_1(t,α), x\_2(t,α)], and the α-cut of ỹ(t) is defined as [y\_1(t,α), y\_2(t,α)]. Here, α is a parameter that satisfies the condition 0 ≤ α ≤ 1.

**4.1.** **Case 1: when 𝑥(𝑡) and 𝑦(𝑡) both are i-gH differentiable**

The system of equation (1) can be written as

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)-ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$$

$\frac{dx\_{2}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)-ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$(2)

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{1}\left(t,α\right)y\_{2}\left(t,α\right)-d\_{2}y\_{2}\left(t,α\right)$

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{2}(t,α)y\_{1}(t,α)-d\_{2}y\_{1}(t,α)$$

With$ x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

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**4.1.1.Stability analysis of system**

 Interior point equilibrium of the system of equation (2) is given by$E\_{1}^{\*}(x\_{1}^{1\*},x\_{2}^{1\*},y\_{1}^{1\*},y\_{2}^{1\*})$

 Where,

 $x\_{1}^{1\*}=\frac{d\_{2}}{ka}$, $x\_{2}^{1\*}=\frac{d\_{2}}{ka}$,

 $y\_{1}^{1\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}},$ $y\_{2}^{1\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

The existance condition of the equilibrium point $E\_{1}^{\*}$ is $kk\_{1}a(r-e)>d\_{2}r$. This condition is also feasible.

Jacobian matrix corresponding to the equilibrium point $E\_{1}^{\*}$ is given by $V\_{1}^{\*}$,

Where, $V\_{1}^{\*}= \left(\begin{matrix}\begin{matrix}0\\a\_{21}\\\begin{matrix}a\_{31}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{12}\\0\\\begin{matrix}0\\a\_{42}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{13}\\0\\\begin{matrix}0\\0\end{matrix}\end{matrix}&\begin{matrix}0\\a\_{24}\\\begin{matrix}0\\0\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right)$

And,

 $a\_{12}=r-\frac{2rx\_{2}^{1\*}}{k\_{1}}-ay\_{1}^{1\*}-e$, $a\_{13}=-ax\_{2}^{1\*},$

 $a\_{21}=r-\frac{2rx\_{1}^{1\*}}{k\_{1}}-ay\_{2}^{1\*}-e,$ $a\_{24}= -ax\_{1}^{1\*},$

 $a\_{31}= kay\_{2}^{1\*},$

 $a\_{42}= kay\_{1}^{1\*}$

The characteristic equation is given by

 $ξ\_{1}^{4}+A\_{1}ξ\_{1}^{3}+A\_{2}ξ\_{1}^{2}+A\_{3}ξ\_{1}+A\_{4}=0$

Where,

$$A\_{1}=0.$$

$A\_{2}=a\_{12}a\_{21}+a\_{24}a\_{42}+a\_{13}a\_{31}$ . where, $a\_{12}=r-\frac{2rx\_{2}^{1\*}}{k\_{1}}-ay\_{1}^{1\*}-e$ ,$ a\_{13}=-ax\_{2}^{1\*},a\_{21}=r-\frac{2rx\_{1}^{1\*}}{k\_{1}}-ay\_{2}^{1\*}-e,a\_{24}= -ax\_{1}^{1\*},a\_{24}= -ax\_{1}^{1\*},a\_{31}= kay\_{2}^{1\*},a\_{42}= kay\_{1}^{1\*}$.

$A\_{3}=$ 0.

$A\_{4}=a\_{13}a\_{24}a\_{31}a\_{42}$.

Here$A\_{1}=$ 0$, A\_{3}=$ 0,

Therefore according to the Routh-Hurwitz criteria, the above system is unstable.

So we can conclude that, the system is unstable if 𝑥(𝑡) and𝑦(𝑡) both are i-gH differentiable.

**4.2.Case 2: when 𝑥(𝑡) is i-gH differentiable and 𝑦(𝑡) is ii-gH differentiable**

The system of equation (1) transformed to the model

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)- ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$$

$$\frac{dx\_{2}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)- ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$$

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{2}\left(t,α\right)y\_{1}\left(t,α\right)-d\_{2}y\_{1}\left(t,α\right)$(3)

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{1}(t,α)y\_{2}(t,α)-d\_{2}y\_{2}(t,α)$$

With $x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

**4.2.1. Stability analysis of system**

 The interior point equilibrium corresponding to the system of equation (6) is given by $E\_{2}^{\*}(x\_{1}^{2\*},x\_{2}^{2\*},y\_{1}^{2\*},y\_{2}^{2\*})$

 Where,

 $x\_{1}^{2\*}=\frac{d\_{2}}{ka }$, $x\_{2}^{2\*}=\frac{d\_{2}}{ka}$,

 $y\_{1}^{2\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}},$ $y\_{2}^{2\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

The existance condition of the equilibrium point $E\_{2}^{\*}$ is $kk\_{1}a(r-e)>d\_{2}r$. This condition is also feasible.

Jacobian matrix corresponding to the equilibrium point $E\_{2}^{\*}$ is given by $V\_{2}^{\*}$,

Where,

$$V\_{2}^{\*}= \left(\begin{matrix}\begin{matrix}0\\a\_{21}\\\begin{matrix}0\\a\_{41}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{12}\\0\\\begin{matrix}a\_{32}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}a\_{13}\\0\\\begin{matrix}0\\0\end{matrix}\end{matrix}&\begin{matrix}0\\a\_{24}\\\begin{matrix}0\\0\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right)$$

and,

 $a\_{12}=r-\frac{2rx\_{2}^{2\*}}{k\_{1}}-ay\_{1}^{2\*}-e$, $a\_{13}=-ax\_{2}^{2\*},$

$a\_{21}=r-\frac{2rx\_{1}^{2\*}}{k\_{1}}-ay\_{2}^{2\*}-e,$ $a\_{24}= -ax\_{1}^{2\*},$

 $a\_{32}= kay\_{1}^{2\*},$

 $a\_{41}= kay\_{2}^{2\*}$.

The corresponding characteristic equation is given as follows

 $ξ\_{2}^{4}+B\_{1}ξ\_{2}^{3}+B\_{2}ξ\_{2}^{2}+B\_{3}ξ\_{2}+B\_{4}=0$

Where,

$$B\_{1}=0,$$

$B\_{2}=a\_{12}a\_{21}.$ where, $a\_{12}=r-\frac{2rx\_{2}^{2\*}}{k\_{1}}-ay\_{1}^{2\*}-e,a\_{21}=r-\frac{2rx\_{1}^{2\*}}{k\_{1}}-ay\_{2}^{2\*}-e,$

$B\_{3}=a\_{12 }a\_{24}a\_{41}+a\_{13}a\_{21}a\_{32}$,

$B\_{4}=a\_{13}a\_{24}a\_{32}a\_{41}$.

Here $B\_{1}=$ 0 ,

 Therefore according to the Routh-Hurwitz criteria, the system is unstable.

 Therefore, we can conclude that, the system is unstable, if 𝑥(𝑡) is i-gH differentiable and 𝑦(𝑡) is ii-gH differentiable.

**4.3. Case 3:When 𝑥(𝑡) is ii-gH differentiable and 𝑦(𝑡) is i-gH differentiable**

The system of equation (1) can be written as

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)-ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$$

$$\frac{dx\_{2}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)-ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$$

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{1}\left(t,α\right)y\_{2}\left(t,α\right)-d\_{2}y\_{2}\left(t,α\right)$(4)

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{2}(t,α)y\_{1}(t,α)-d\_{2}y\_{1}(t,α)$$

With$ x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

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**4.3.1. Stability analysis of system**

 The interior point equilibrium corresponding to system of equation (4) is given by $E\_{3}^{\*}(x\_{1}^{3\*},x\_{2}^{3\*},y\_{1}^{3\*},y\_{2}^{3\*})$

 Where,

 $x\_{1}^{3\*}=\frac{d\_{2}}{ka}$ , $x\_{2}^{3\*}=\frac{d\_{2}}{ka}$ ,

 $y\_{1}^{3\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$ $, y\_{2}^{3\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

The existance condition of the equilibrium point $E\_{3}^{\*}$ is given by $kk\_{1}a(r-e)>d\_{2}r$. This condition is also feasible.

Jacobian matrix corresponding to the equilibrium point $E\_{3}^{\*}$ is given by $V\_{3}^{\*}$,

Where,

$$V\_{3}^{\*}=\left[\begin{matrix}\begin{matrix}a\_{11}\\0\\\begin{matrix}a\_{31}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{22}\\\begin{matrix}0\\a\_{42}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{23}\\\begin{matrix}0\\0\end{matrix}\end{matrix}&\begin{matrix}a\_{14}\\0\\\begin{matrix}0\\0\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right]$$

Where,

 $a\_{11}= r-\frac{2rx\_{1}^{3\*}}{k\_{1}}-ay\_{2}^{3\*}-e$, $a\_{14}=-ax\_{1}^{3\*},$

 $a\_{22}=r-\frac{2rx\_{2}^{3\*}}{k\_{1}}-ay\_{1}^{3\*}-e,$ $a\_{23}= -ax\_{2}^{3\*},$

 $a\_{31}= kay\_{2}^{3\*},$

 $a\_{42}= kay\_{1}^{3\*}$

The characteristic equation is given by

 $ξ\_{3}^{4}+C\_{1}ξ\_{3}^{3}+C\_{2}ξ\_{3}^{2}+C\_{3}ξ\_{3}+C\_{4}=0$

Where,

$C\_{1}=a\_{11}+ a\_{22} .$ where, $a\_{11}= r-\frac{2rx\_{1}^{3\*}}{k\_{1}}-ay\_{2}^{3\*}-e$ , $a\_{22}=r-\frac{2rx\_{2}^{3\*}}{k\_{1}}-ay\_{1}^{3\*}-e$.

$C\_{2}=a\_{11}a\_{22}$ . where, $a\_{11}= r-\frac{2rx\_{1}^{3\*}}{k\_{1}}-ay\_{2}^{3\*}-e$ , $a\_{22}=r-\frac{2rx\_{2}^{3\*}}{k\_{1}}-ay\_{1}^{3\*}-e$.

$C\_{3}=$ 0.

$C\_{4}=a\_{14}a\_{23}a\_{31}a\_{42}$. Where $a\_{14}=-ax\_{1}^{3\*},a\_{23}= -ax\_{2}^{3\*},a\_{31}= kay\_{2}^{3\*},a\_{42}= kay\_{1}^{3\*}$.

Here$C\_{3}=$ 0,

Therefore the above system is unstable by Routh-Hurwitz criteria,

So we can say that when 𝑥(𝑡) is ii-gH differentiable and 𝑦(𝑡) is i-gH differentiable, the system is unstable.

**4.4.Case 4:when 𝑥(𝑡) and 𝑦(𝑡) both are ii-gH differentiable**

The system of equation (1) can be written as

$$\frac{dx\_{1}(t,α)}{dt}=rx\_{1}(t,α)\left(1-\frac{x\_{1}(t,α)}{k\_{1}}\right)- ax\_{1}(t,α)y\_{2}(t,α)-ex\_{1}(t,α)$$

$\frac{dx\_{2}(t,α)}{dt}=rx\_{2}(t,α)\left(1-\frac{x\_{2}(t,α)}{k\_{1}}\right)- ax\_{2}(t,α)y\_{1}(t,α)-ex\_{2}(t,α)$ (5)

$\frac{dy\_{1}\left(t,α\right)}{dt}=kax\_{2}\left(t,α\right)y\_{1}\left(t,α\right)-d\_{2}y\_{1}\left(t,α\right)$

$$\frac{dy\_{2}(t,α)}{dt}=kax\_{1}(t,α)y\_{2}(t,α)-d\_{2}y\_{2}(t,α)$$

With $x\_{1}\left(0,α\right)=x\_{01}\left(α\right),x\_{2}\left(0,α\right)=x\_{02}\left(α\right), y\_{1}\left(0,α\right)=y\_{01}\left(α\right) and y\_{2}\left(0,α\right)=y\_{02}\left(α\right)$

**4.4.1. Stability analysis of system**

The interior point equilibrium of system of equation (5) is given by $E\_{4}^{\*}(x\_{1}^{4\*},x\_{2}^{4\*},y\_{1}^{4\*},y\_{2}^{4\*})$

Where,

 $x\_{1}^{4\*}=\frac{d\_{2}}{ka}$, $x\_{2}^{4\*}=\frac{d\_{2}}{ka}$,

 $y\_{1}^{4\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}},$ $y\_{2}^{4\*}= \frac{kk\_{1}a(r-e)-d\_{2}r}{kk\_{1}a^{2}}$

The existance condition of the equilibrium point $E\_{4}^{\*}$ is $kk\_{1}a(r-e)>d\_{2}r$. This condition id feasible condition also.

The corresponding jacobian matrix is given by $V\_{4}^{\*}$,

$$V\_{4}^{\*}= \left[\begin{matrix}\begin{matrix}a\_{11}\\0\\\begin{matrix}0\\a\_{41}\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{22}\\\begin{matrix}a\_{32}\\0\end{matrix}\end{matrix}&\begin{matrix}\begin{matrix}0\\a\_{23}\\\begin{matrix}a\_{33}\\0\end{matrix}\end{matrix}&\begin{matrix}a\_{14}\\0\\\begin{matrix}0\\a\_{44}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\end{matrix}\right]$$

Where,

 $a\_{11}= r-\frac{2rx\_{1}^{4\*}}{k\_{1}}-ay\_{2}^{4\*}-e$, $a\_{14}=-ax\_{1}^{4\*},$

 $a\_{22}=r-\frac{2rx\_{2}^{4\*}}{k\_{1}}-ay\_{1}^{4\*}-e,$ $a\_{23}= -ax\_{2}^{4\*},$

 $a\_{32}= kay\_{1}^{4\*},$ $a\_{33} = 0,$

 $a\_{41}= kay\_{2}^{4\*}$, $a\_{44} = 0.$

The characteristic equation is given by

 $ξ^{\*4}+P\_{1}ξ^{\*3}+P\_{2}ξ^{\*2}+P\_{3}ξ^{\*}+P\_{4}=0$

Where,

$P\_{1}=a\_{11}+ a\_{22} .$ where, $a\_{11}= r-\frac{2rx\_{1}^{\*4}}{k\_{1}}-ay\_{2}^{\*4}-e$,$a\_{22}=r-\frac{2rx\_{2}^{\*4}}{k\_{1}}-ay\_{1}^{\*4}-e,$

$P\_{2}=a\_{23}a\_{32}-a\_{11}a\_{22}+a\_{14}a\_{41}$ . where, $a\_{14}=-ax\_{1}^{\*4},a\_{23}= -ax\_{2}^{\*4}, a\_{32}= kay\_{1}^{\*4},a\_{41}= kay\_{2}^{\*4}$.

$P\_{3}=a\_{11}a\_{23}a\_{32}+a\_{14}a\_{22}a\_{41}$,

$P\_{4}=a\_{14}a\_{23}a\_{32}a\_{41}$.

Now,

$P\_{1}>$0

$P\_{2}>$0

$P\_{3}>$0

$P\_{4}>$0

The system of equation (5) is stable if $P\_{1},P\_{2},P\_{3},P\_{4}$ all ae positive and $P\_{1}P\_{2}-P\_{3}>0$ and $P\_{1}P\_{2}P\_{3}-P\_{3}^{2}-P\_{1}^{2}P\_{4}>0, $according to the Routh-Hurwitz criteria.

So we can say that, when 𝑥(𝑡) and 𝑦(𝑡) both are ii-gH differentiable, the system is stable.

1. **DISCUSSION AND CONCLUSION**

In this research article, we investigate the dynamic behavior of predator-prey interactions with harvesting within a fuzzy environment. In mathematical representations of ecological problems, not all parameters need to be precise; some may exhibit imprecision to capture various behavioral aspects. The presence of impreciseness among these parameters leads to changes in the system's dynamical behavior. Fuzzy concepts play a crucial role in understanding and modeling such dynamical systems.

In this research article at first we consider a two diamentional Predator-Prey system with harvesting then we establish four set of fuzzy differential equation model of Predator-Prey system with harvesting together with fuzzy initial value. In the fuzzy differential equation at first we consider both the population i.e. 𝑥(𝑡, 𝛼), 𝑦(𝑡, 𝛼) are i-gH differentiable and observe the dynamical behavior of the model analytically, then we consider that 𝑥(𝑡, 𝛼) is i-gH differentiable and 𝑦(𝑡, 𝛼) is ii-gH differentiable and observe the dynamical behavior of the model, then we consider that 𝑥(𝑡, 𝛼) is ii-gH differentiable and 𝑦(𝑡, 𝛼) is i-gH differentiable and observe the dynamical behavior of the model analytically. At last we consider both the population 𝑥(𝑡, 𝛼), 𝑦(𝑡, 𝛼) are ii-gH differentiable and observe the dynamical behavior of the model analytically. Here we observe that the equilibrium point corresponding to the first three fuzzy system of equation are unstable and the equilibrium point corresponding to the last fuzzy system of equation is stable i.e when both the population is ii-gH differentiable then the solution is stable.

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