**Product Signed Dominating Function**

**T. M. Velammal, Research Scholar (Reg. No. 21212232092010)**

PG & Research Department of Mathematics

V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India

avk.0912@gmail.com

**A. Nagarajan, Head & Associate Professor (Retd.)**

PG & Research Department of Mathematics

V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India

nagarajan.voc@gmail.com

**K. Palani, Head & Associate Professor**

PG & Research Department of Mathematics

A.P.C. Mahalaxmi College For Women, Thoothukudi-628002, Tamil Nadu, India

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.

palani@apcmcollege.ac.in

**ABSTRACT:**

Let $G=(V,E)$ be a simple graph. A function $f:V\rightarrow \left\{-1,1\right\}$ is called a product signed dominating function, if $ f\left[v\right]=1 ∀ v\in V$ where $f\left[v\right]=\prod\_{u\in N[v]}^{}f(u)$ and $N[v]$ denotes the closed neighborhood of $v$. The weight $f(G)$ of a function $f$ is defined as $f\left(G\right)=\sum\_{v\in V}^{}f(v)$. The minimum positive weight of a product signed dominating function is called product signed domination number of a graph $G$ and is denoted by$ γ\_{sign}^{\*}\left(G\right).$ In this paper, we discuss product signed dominating functions for some special graphs.

**Keywords:** Fan graph, wheel graph, helm graph, flower graph, product signed dominating function, product signed domination number.

**AMS Subject Classification:** 05C69.

**I. INTRODUCTION**

The domination problem was studied from 1950s onwards. Richard Karp proved the set cover problem to be NP-complete which had implications for the dominating set problem. Dunbar et al. introduced signed domination number [2],[3],[4],[5]. The concept of product signed domination was introduced in [11]. Hereafter, we denote the weight of a graph $G$ with respect to the function $f$ as $w\_{f}\left(G\right).$ Definitions of fan graph, wheel graph and helm graph are from [1]. Seoud and Youssef defined flower graph in [1]. In this paper, we find product signed domination number for fan graph, wheel graph, helm graph and flower graph.

**II. Main Results**

**2.1 Theorem**

For $n\geq 3,$ $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=\left\{\begin{array}{c}\frac{n-8}{3} if n≡2\left(mod 6\right) and n>8\\\frac{n-6}{3} if n≡3\left(mod 6\right) and n>8\\\frac{n-4}{3} if n≡4\left(mod 6\right) and n>8\\n otherwise \end{array}\right.$

**Proof:**

Let $F\_{1,n-1}$ be a fan graph on $n$ vertices.

Let $V=\left\{v\_{1},v\_{2},…,v\_{n-1},v\right\}$ and $E=\left\{v\_{b}v\_{b+1}|1\leq b\leq n-2\right\}∪\left\{vv\_{b}|1\leq b\leq n-1\right\}$

**Case 1:**

$$f\left(v\right)=1$$

**Subcase 1.1:**

If $f\left(v\_{1}\right)= -1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= -1$

Again to get $f\left[v\_{2}\right]as 1$, set $f\left(v\_{3}\right)=1$

Proceeding like this, we define $f:V\left(F\_{1,n-1}\right)\rightarrow \{-1,+1\}$ as

For $1\leq b\leq n, f\left(v\_{b}\right)=\left\{\begin{array}{c}1 if b≡0(mod 3)\\-1 otherwise\end{array}\right.$

This $f$ may be a product signed dominating function. If it is, the weight will be negative since $N\_{f}[-1]>N\_{f}[1]$. **[11]**

**Subcase 1.2:**

If $f\left(v\_{1}\right)= 1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= 1$

Again to get $f\left[v\_{2}\right] as 1$, set $f\left(v\_{3}\right)=1$

Proceeding like this, we have $f\left(v\_{4}\right)=f\left(v\_{5}\right)=…=f\left(v\_{n-1}\right)=1$

In this case the weight is $n$, the total number of vertices,

**Case 2:**

$$f\left(v\right)=-1$$

For $2\leq b\leq n-2$, it is observed that

If $f\left(v\_{b}\right)=-1$ then $2$ cases arise

(i) if $f\left(v\_{b-1}\right)=-1$, then $f\left(v\_{b+1}\right)=-1$

(ii) if $f\left(v\_{b-1}\right)=1$, then $f\left(v\_{b+1}\right)=1$

And if $f\left(v\_{b}\right)=1$ then $2$ cases arise

(i) if $f\left(v\_{b-1}\right)=-1$, then $f\left(v\_{b+1}\right)=1$

(ii) if $f\left(v\_{b-1}\right)=1$, then $f\left(v\_{b+1}\right)=-1$

**Subcase 2.1:**

If $f\left(v\_{1}\right)= 1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= -1$

Again to get $f\left[v\_{2}\right] as 1$, set $f\left(v\_{3}\right)=1$

Again to get $f\left[v\_{3}\right] as 1$, set $f\left(v\_{4}\right)=1$

Proceeding like this, we define $f:V\left(F\_{1,n-1}\right)\rightarrow \{-1,+1\}$ as

For $1\leq b\leq n, f\left(v\_{b}\right)=\left\{\begin{array}{c}-1 if b≡2(mod 3)\\1 otherwise\end{array}\right.$

**Subcase 2.2:**

If $f\left(v\_{1}\right)= -1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= 1$

Again to get $f\left[v\_{2}\right] as 1$, set $f\left(v\_{3}\right)=1$

Again to get $f\left[v\_{3}\right] as 1$, set $f\left(v\_{4}\right)=-1$

Again to get $f\left[v\_{4}\right] as 1$, set $f\left(v\_{5}\right)=1$

Proceeding like this, we define $f:V\left(F\_{1,n-1}\right)\rightarrow \{-1,+1\}$ as

For $1\leq b\leq n, f\left(v\_{b}\right)=\left\{\begin{array}{c}-1 if b≡1(mod 3)\\1 otherwise\end{array}\right.$

**When** $n=3$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n=3$.By subcase 2.1, $w\_{f}\left(F\_{1,n-1}\right)=-1$, a negative integer.By subcase 2.2, $w\_{f}\left(F\_{1,n-1}\right)=-1$, a negative integer.Therefore, $γ\_{sign}^{\*}\left(F\_{1,2}\right)=3.$

**When** $n=4$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n=4$. By subcase 2.1, $w\_{f}\left(F\_{1,n-1}\right)=0$. By subcase 2.2, $f\left[v\_{3}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,3}\right)=4.$

**When** $n=5$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n=5$. By subcase 2.1, $f\left[v\_{4}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,4}\right)=5.$

**When** $n=6$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n=6$. By subcases 2.1 and 2.2, $f\left[v\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,5}\right)=6.$

**When** $n=7$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n=7$. By subcase 2.1, $f\left[v\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=f\left[v\_{6}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,6}\right)=7.$

**When** $n=8$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n=8$. By subcase 2.1, $f\left[v\right]=f\left[v\_{7}\right]=-1\ne 1$. By subcase 2.2, $w\_{f}\left(F\_{1,n-1}\right)=0$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,7}\right)=8.$

**Consider** $n>8$

**For** $n≡0(mod 6)$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n$. By subcases 2.1 and 2.2, $f\left[v\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=n.$

**For** $n≡1(mod 6)$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n$. By subcase 2.1, $f\left[v\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=f\left[v\_{n-1}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=n.$

**For** $n≡2(mod 6)$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n$. By subcase 2.1, $f\left[v\right]=f[v\_{n-1}]-1\ne 1$. By subcase 2.2, $w\_{f}\left(F\_{1,n-1}\right)=\frac{n-8}{3}$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=min\left\{n,\frac{n-8}{3}\right\}=\frac{n-8}{3}.$

**For** $n≡3(mod 6)$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n$. By subcase 2.1, $w\_{f}\left(F\_{1,n-1}\right)=\frac{n-6}{3}$. By subcase 2.2, $w\_{f}\left(F\_{1,n-1}\right)=\frac{n-6}{3}$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=min\left\{n,\frac{n-6}{3},\frac{n-6}{3}\right\}=\frac{n-6}{3}.$

**For** $n≡4(mod 6)$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n$. By subcase 2.1, $w\_{f}\left(F\_{1,n-1}\right)=\frac{n-4}{3}$. By subcase 2.2, $f\left[v\_{n-1}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=min\left\{n,\frac{n-4}{3}\right\}=\frac{n-4}{3}.$

**For** $n≡5(mod 6)$

By subcase 1.2, $w\_{f}\left(F\_{1,n-1}\right)=n$. By subcase 2.1, $f\left[v\_{n-1}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=n.$

 Also from the above discussion, it is clear that, by subcase 2.1, $f$ is not a product signed dominating function when $n≡0,1,2,5(mod 6)$ and by subcase 2.2, $f$ is not a product signed dominating function when $n≡0,1,4,5(mod 6)$

Therefore, $ γ\_{sign}^{\*}\left(F\_{1,n-1}\right)=\left\{\begin{array}{c}\frac{n-8}{3} if n≡2\left(mod 6\right) and n>8\\\frac{n-6}{3} if n≡3\left(mod 6\right) and n>8\\\frac{n-4}{3} if n≡4\left(mod 6\right) and n>8\\n otherwise \end{array}\right.$

**2.2 Illustration**

$$v$$

$$v\_{11}$$

$$v\_{10}$$

$$v\_{9}$$

$$v\_{8}$$

$$v\_{7}$$

$$v\_{6}$$

$$v\_{5}$$

$$v\_{4}$$

$$v\_{3}$$

$$v\_{2}$$

$$v\_{1}$$

$$-1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$v\_{12}$$

$$1$$

$$v\_{13}$$

$$-1$$

**Figure 1**

**Product signed dominating function for fan graph on** $n=14≡2(mod 6)$ **vertices.**

$γ\_{sign}^{\*}\left(F\_{1,13}\right)=\frac{14-8}{3}=2$.

**2.3 Illustration**

$$v$$

$$v\_{1}$$

$$v\_{2}$$

$$v\_{3}$$

$$v\_{4}$$

$$v\_{5}$$

$$v\_{6}$$

$$v\_{7}$$

$$v\_{8}$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

**Figure 2**

**Product signed dominating function for fan graph on** $n=9≡3(mod 6)$ **vertices by subcase 2.1 of 2.1**

$$v$$

$$v\_{1}$$

$$v\_{2}$$

$$v\_{3}$$

$$v\_{4}$$

$$v\_{5}$$

$$v\_{6}$$

$$v\_{7}$$

$$v\_{8}$$

$$-1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

**Figure 3**

**Product signed dominating function for fan graph on** $n=9≡3(mod 6)$ **vertices by subcase 2.2 of 2.1**

By subcase 2.1 of 2.1, $w\_{f}\left(F\_{1,8}\right)=\frac{9-6}{3}=1$. By subcase 2.2 of 2.1, $w\_{f}\left(F\_{1,8}\right)=\frac{9-6}{3}=1$. Therefore, $γ\_{sign}^{\*}\left(F\_{1,8}\right)=1.$

**2.4 Illustration**

$$v$$

$$v\_{1}$$

$$v\_{2}$$

$$v\_{3}$$

$$v\_{4}$$

$$v\_{5}$$

$$v\_{6}$$

$$v\_{7}$$

$$v\_{8}$$

$$v\_{9}$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

**Figure 4**

**Product signed dominating function for fan graph on** $n=10≡4(mod 6)$ **vertices.**

$γ\_{sign}^{\*}\left(F\_{1,9}\right)=\frac{10-4}{3}=2$.

**2.5 Theorem**

For $n\geq 4,$ $γ\_{sign}^{\*}\left(W\_{n}\right)=\left\{\begin{array}{c}\frac{n-4}{3} if n≡4\left(mod 6\right) and n>4\\n otherwise\end{array}\right.$

**Proof:**

Let $W\_{n}$ represent a wheel graph on $n$ vertices.

Let $V=\left\{v\_{1},v\_{2},…,v\_{n-1},v\right\}$ and $E=\left\{v\_{b}v\_{b+1}|1\leq b\leq n-2\right\}∪\left\{vv\_{b}|1\leq b\leq n-1\right\}∪\{v\_{1}v\_{n-1}\}$

**Case 1:**

$$f\left(v\right)=1$$

**Subcase 1.1:**

If $f\left(v\_{1}\right)= -1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= -1$

Again to get $f\left[v\_{2}\right]as 1$, set $f\left(v\_{3}\right)=1$

Proceeding like this, we define $f:V\left(W\_{n}\right)\rightarrow \{-1,+1\}$ as

For $1\leq b\leq n, f\left(v\_{i}\right)=\left\{\begin{array}{c}1 if b≡0(mod 3)\\-1 otherwise\end{array}\right.$

This $f$ may be a product signed dominating function. If it is, the weight will be negative since $N\_{f}[-1]>N\_{f}[1]$. **[11]**

**Subcase 1.2:**

If $f\left(v\_{1}\right)= 1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= 1$

Again to get $f\left[v\_{2}\right] as 1$, set $f\left(v\_{3}\right)=1$

Proceeding like this, we have $f\left(v\_{4}\right)=f\left(v\_{5}\right)=…=f\left(v\_{n-1}\right)=1$

In this case the weight is $n$, the total number of vertices,

**Case 2:**

$$f\left(v\right)=-1$$

For $2\leq b\leq n-2$, it is observed that

If $f\left(v\_{b}\right)=-1$ then $2$ cases arise

(i) if $f\left(v\_{b-1}\right)=-1$, then $f\left(v\_{b+1}\right)=-1$

(ii) if $f\left(v\_{b-1}\right)=1$, then $f\left(v\_{b+1}\right)=1$

And if $f\left(v\_{b}\right)=1$ then $2$ cases arise

(i) if $f\left(v\_{b-1}\right)=-1$, then $f\left(v\_{b+1}\right)=1$

(ii) if $f\left(v\_{b-1}\right)=1$, then $f\left(v\_{b+1}\right)=-1$

**Subcase 2.1:**

If $f\left(v\_{1}\right)=1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)=-1$

Again to get $f\left[v\_{2}\right] as 1$, set $f\left(v\_{3}\right)=1$

Again to get $f\left[v\_{3}\right] as 1$, set $f\left(v\_{4}\right)=1$

Proceeding like this, we define $f:V\left(W\_{n}\right)\rightarrow \{-1,+1\}$ as

For $1\leq b\leq n, f\left(v\_{b}\right)=\left\{\begin{array}{c}-1 if b≡2(mod 3)\\1 otherwise\end{array}\right.$

**Subcase 2.2:**

If $f\left(v\_{1}\right)= -1$, to get $f\left[v\_{1}\right]=1$, set $f\left(v\_{2}\right)= 1$

Again to get $f\left[v\_{2}\right] as 1$, set $f\left(v\_{3}\right)=1$

Again to get $f\left[v\_{3}\right] as 1$, set $f\left(v\_{4}\right)=-1$

Again to get $f\left[v\_{4}\right] as 1$, set $f\left(v\_{5}\right)=1$

Proceeding like this, we define $f:V\left(W\_{n}\right)\rightarrow \{-1,+1\}$ as

For $1\leq b\leq n, f\left(v\_{b}\right)=\left\{\begin{array}{c}-1 if b≡1(mod 3)\\1 otherwise\end{array}\right.$

**When** $n=4$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n=4$. By subcases 2.1 and 2.2, $w\_{f}\left(W\_{n}\right)=0$. Therefore, $γ\_{sign}^{\*}\left(W\_{4}\right)=4.$

**When** $n=5$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n=5$. By subcase 2.1, $f\left[v\right]=f\left[v\_{1}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=f\left[v\_{1}\right]=f\left[v\_{4}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(W\_{5}\right)=5.$

**Consider** $n\geq 6$

**For** $n≡0(mod 6)$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n$. By subcase 2.1, $f\left[v\right]=f\left[v\_{1}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=f\left[v\_{n-1}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(W\_{n}\right)=n.$

**For** $n≡1(mod 6)$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n$. By subcases 2.1 and 2.2, $f\left[v\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(W\_{n}\right)=n.$

**For** $n≡2(mod 6)$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n$. By subcase 2.1, $f\left[v\right]=f\left[v\_{n-1}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\_{1}\right]=f\left[v\_{n-1}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(W\_{n}\right)=n.$

**For** $n≡3(mod 6)$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n$. By subcase 2.1, $f\left[v\_{1}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\_{n-1}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(W\_{n}\right)=n.$

**For** $n≡4(mod 6)$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n$. By subcases 2.1 and 2.2, $w\_{f}\left(W\_{n}\right)=\frac{n-4}{3}$. Therefore, $γ\_{sign}^{\*}\left(W\_{n}\right)=min\left\{n,\frac{n-4}{3},\frac{n-4}{3}\right\}=\frac{n-4}{3}.$

**For** $n≡5(mod 6)$

By subcase 1.2, $w\_{f}\left(W\_{n}\right)=n$. By subcase 2.1, $f\left[v\_{n-1}\right]=-1\ne 1$. By subcase 2.2, $f\left[v\right]=f\left[v\_{1}\right]=f\left[v\_{n-1}\right]=-1\ne 1$. Therefore, $γ\_{sign}^{\*}\left(W\_{n}\right)=n.$

Also from the above discussion, it is clear that the functions defined in subcases 2.1 and 2.2 are not product signed dominating functions when $n≡0,1,2,3,5(mod 6)$

Therefore, $ γ\_{sign}^{\*}\left(W\_{n}\right)=\left\{\begin{array}{c}\frac{n-4}{3} if n≡4\left(mod 6\right) and n>4\\n otherwise\end{array}\right.$

**2.6 Illustration**

$$1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$-1$$

$$-1$$

$$v$$

$$v\_{7}$$

$$v\_{6}$$

$$v\_{5}$$

$$v\_{4}$$

$$v\_{3}$$

$$v\_{9}$$

$$v\_{8}$$

$$v\_{2}$$

$$v\_{1}$$

**Figure 5**

**Product signed dominating function for fan graph on** $n=10≡4(mod 6)$ **vertices by subcase 2.1 of 2.5**

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$1$$

$$-1$$

$$v$$

$$v\_{7}$$

$$v\_{6}$$

$$v\_{5}$$

$$v\_{4}$$

$$v\_{3}$$

$$v\_{9}$$

$$v\_{8}$$

$$v\_{2}$$

$$v\_{1}$$

**Figure 6**

**Product signed dominating function for fan graph on** $n=10≡4(mod 6)$ **vertices by subcase 2.2 of 2.5**

By subcase 2.1 of 2.5, $w\_{f}\left(W\_{10}\right)=\frac{10-4}{3}=2$.

By subcase 2.2 of 2.5, $w\_{f}\left(W\_{10}\right)=\frac{10-4}{3}=2$.

Therefore, $γ\_{sign}^{\*}\left(W\_{10}\right)=2.$

**2.7 Theorem:**

Let $n>3$ be any integer and $G≅H\_{n}$, a helm graph on $2n-1$ vertices. Then

$$γ\_{sign}^{\*}\left(G\right)=\left\{\begin{array}{c}1 when n≡1(mod4)\\2n-1 otherwise\end{array}\right.$$

**Proof:**

Let $V\left(G\right)=\{v,v\_{b},u\_{b}|1\leq b\leq n-1\}$ with $u\_{b},1\leq b\leq n-1$ as the pendant vertices and $E\left(G\right)=\left\{1\leq b\leq n-1\right\}∪\left\{1\leq b\leq n-2\right\}∪\left\{v\_{1}v\_{n-1}\right\}∪\{v\_{b}u\_{b}|1\leq b\leq n-1\}$

Here $v\_{b}$ and $u\_{b}$ where $1\leq b\leq n-1$ must be assigned the same functional value **[11]**.

**Let** $f\left(v\right)=-1$**.**

To get $f\left[v\right]$ as $1$, odd number of $v\_{b}'$s where $1\leq b\leq n-1 $must be assigned $-1$.

**Suppose** $n-1$ **is even,**

Assign $-1$ to $v\_{1},v\_{2},…,v\_{n-2}$ and take $f\left(v\_{n-1}\right)=1$. Correspondingly, $f\left(u\_{b}\right)=-1$ for $1\leq b\leq n-2$ and $f\left(u\_{n-1}\right)=1$.

Now $f\left[v\right]=1$ obviously.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(-1)(-1)$$

 $=-1$

Hence $f$ is not a product signed dominating function.

 Assign $-1$ to $v\_{1},v\_{2},…,v\_{n-4}$ and $1$ to $v\_{n-3},v\_{n-2},v\_{n-1}$

Correspondingly, $f\left(u\_{b}\right)=\left\{\begin{array}{c}-1, 1\leq b\leq n-4\\1 otherwise\end{array}\right.$

Here also $f\left[v\right]=1$ obviously.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-2}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(v\_{n-2}\right)f\left(u\_{n-2}\right)f(v\_{n-3})$$

$$ =(-1)(1)(1)(1)(1)$$

 $=-1$

Hence $f$ is not a valid product signed dominating function.

 Assign $-1$ to $v\_{1},v\_{2},…,v\_{n-6}$ and $1$ to $v\_{n-5},v\_{n-4},v\_{n-3},v\_{n-2},v\_{n-1}$

Correspondingly, $f\left(u\_{b}\right)=\left\{\begin{array}{c}-1, 1\leq b\leq n-6\\1 otherwise\end{array}\right.$

Clearly, here also $f\left[v\right]=1$.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-2}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(v\_{n-2}\right)f\left(u\_{n-2}\right)f(v\_{n-3})$$

$$ =(-1)(1)(1)(1)(1)$$

 $=-1$

Hence $f$ is not a valid product signed dominating function.

Continuing like this,

 Assign $-1$ to $v\_{1}$ and $1$ to $v\_{b}$ where $2\leq b\leq n-1$

Correspondingly, $f\left(u\_{b}\right)=\left\{\begin{array}{c}-1 if b=1\\1 otherwise\end{array}\right.$

Clearly, $f\left[v\right]=1$.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-2}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(v\_{n-2}\right)f\left(u\_{n-2}\right)f(v\_{n-3})$$

$$ =(-1)(1)(1)(1)(1)$$

 $=-1$

Hence $f$ is not a valid product signed dominating function.

**Suppose** $n-1$ **is odd,**

Assign $-1$ to $v\_{1},v\_{2},…,v\_{n-1}$. Correspondingly, $f\left(u\_{b}\right)=-1$ for $1\leq b\leq n-1$.

Now $f\left[v\right]=1$ obviously.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(-1)(-1)(-1)(-1)$$

 $=-1$

Hence $f$ is not a product signed dominating function.

 Assign $-1$ to $v\_{1},v\_{2},…,v\_{n-4},v\_{n-3}$ and $1$ to $v\_{n-2},v\_{n-1}$

Correspondingly, $f\left(u\_{b}\right)=\left\{\begin{array}{c}-1, 1\leq b\leq n-3\\1 otherwise\end{array}\right.$

Here also $f\left[v\right]=1$ obviously.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-2}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(v\_{n-2}\right)f\left(u\_{n-2}\right)f(v\_{n-3})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-3}]=f\left(v\right)f\left(v\_{n-2}\right)f\left(v\_{n-3}\right)f\left(u\_{n-3}\right)f(v\_{n-4})$$

$$ =(-1)(1)(-1)(-1)(-1)$$

 $=1$

$$f[v\_{n-4}]=f\left(v\right)f\left(v\_{n-3}\right)f\left(v\_{n-4}\right)f\left(u\_{n-4}\right)f(v\_{n-5})$$

$$ =(-1)(-1)(-1)(-1)(-1)$$

 $=1$

Hence $f$ is not a valid product signed dominating function.

 Assign $-1$ to $v\_{1},v\_{2},…,v\_{n-6},v\_{n-5}$ and $1$ to $v\_{n-4},v\_{n-3},v\_{n-2},v\_{n-1}$

Correspondingly, $f\left(u\_{b}\right)=\left\{\begin{array}{c}-1, 1\leq b\leq n-5\\1 otherwise\end{array}\right.$

Clearly, here also $f\left[v\right]=1$.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-2}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(v\_{n-2}\right)f\left(u\_{n-2}\right)f(v\_{n-3})$$

$$ =(-1)(1)(1)(1)(1)$$

 $=-1$

Hence $f$ is not a valid product signed dominating function.

Continuing like this,

 Assign $-1$ to $v\_{1}$ and $1$ to $v\_{b}$ where $2\leq b\leq n-1$

Correspondingly, $f\left(u\_{b}\right)=\left\{\begin{array}{c}-1 if b=1\\1 otherwise\end{array}\right.$

Clearly, $f\left[v\right]=1$.

$$f[v\_{n-1}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(u\_{n-1}\right)f\left(v\_{n-2}\right)f(v\_{1})$$

$$ =(-1)(1)(1)(1)(-1)$$

 $=1$

$$f[v\_{n-2}]=f\left(v\right)f\left(v\_{n-1}\right)f\left(v\_{n-2}\right)f\left(u\_{n-2}\right)f(v\_{n-3})$$

$$ =(-1)(1)(1)(1)(1)$$

 $=-1$

Hence $f$ is not a valid product signed dominating function.

Therefore, assigning $-1$ or $1$ to continuous $v\_{b}'$s fails to give a product signed dominating function.

Redefine $f$ as $f\left(v\right)=-1$ and

$$f\left(v\_{b}\right)=\left\{\begin{array}{c}-1 if b is odd\\1 otherwise\end{array}\right.$$

Correspondingly, $ f\left(u\_{b}\right)=\left\{\begin{array}{c}-1 if b is odd\\1 otherwise\end{array}\right.$

Now $f\left[v\right]=1$ only when $n$ is odd such that $\frac{n-1}{2}$ is odd.

But here, $f\left[v\_{1}\right]=f\left(v\right)f\left(v\_{1}\right)f\left(u\_{1}\right)f\left(v\_{2}\right)f(v\_{n-1})$

 $=(-1)(-1)(-1)(1)(1)$

 $=-1$

Therefore this also does not lead to any product signed dominating function.

Assign $f\left(v\right)=f\left(v\_{1}\right)=-1$. Then $f\left(u\_{1}\right)=-1.$

Correspondingly, $f\left[v\_{1}\right]=f\left(v\right)f\left(v\_{1}\right)f\left(u\_{1}\right)f\left(v\_{2}\right)f(v\_{n-1})$

 $=\left(-1\right)\left(-1\right)\left(-1\right)f\left(v\_{2}\right)f(v\_{n-1})$

 $=1 $if and only if $f\left(v\_{2}\right)$ and $f(v\_{n-1})$ are of opposite sign.

Without loss of generality, assume $f\left(v\_{2}\right)=-1$ and $f\left(v\_{n-1}\right)=1$

Then $f\left(u\_{2}\right)=-1$ and $f\left(u\_{n-1}\right)=1$

Correspondingly, $f\left[v\_{2}\right]=f\left(v\right)f\left(v\_{1}\right)f\left(v\_{3}\right)f\left(v\_{2}\right)f(u\_{2})$

 $=\left(-1\right)\left(-1\right)f\left(v\_{3}\right)(-1)(-1)$

 $=1 $if and only if $f\left(v\_{3}\right)=1$

Let $f\left(v\_{3}\right)=1$. Then $f\left(u\_{3}\right)=1$.

Correspondingly, $f\left[v\_{3}\right]=f\left(v\right)f\left(v\_{3}\right)f\left(u\_{3}\right)f\left(v\_{2}\right)f(v\_{4})$

 $=\left(-1\right)\left(1\right)\left(1\right)(-1)f(v\_{4})$

 $=1 $if and only if $f\left(v\_{4}\right)=1$.

Let $f\left(v\_{4}\right)=1$. Then $f\left(u\_{4}\right)=1$.

Repeating the above procedure, $f\left(v\_{5}\right)=-1, f\left(v\_{6}\right)=-1, f\left(v\_{7}\right)=1, f\left(v\_{8}\right)=1$ and so on. (i.e) $f(v\_{b})$ where $1\leq b\leq n-1$ follows the pattern $-1,-1,1,1$ for every four vertices starting from $v\_{1}$ Therefore, if $n-1=4k,$ then the function is defined by $f\left(v\right)=-1.$

$f\left(v\_{4k+1}\right)=f\left(v\_{4k+2}\right)=-1$ and $f\left(v\_{4k+3}\right)=f\left(v\_{4(k+1)}\right)=1$ for all $k=0 $to $\frac{n-5}{4}$ Correspondingly, $f\left(u\_{4k+1}\right)=f\left(u\_{4k+2}\right)=-1$ and $f\left(u\_{4k+3}\right)=f\left(u\_{4(k+1)}\right)=1$ for all $k=0 $to $\frac{n-5}{4}$

Now by construction, $f\left[v\_{b}\right]=1 ∀ 1\leq b\leq n-2.$

$$f\left[v\_{n-1}\right]=f\left(v\_{n-2}\right)f\left(v\_{n-1}\right)f\left(v\_{1}\right)f\left(u\_{n-1}\right)f(v)$$

 $=(1)(1)(-1)(1)(-$1)

 $=1$

Also by construction, $f\left[u\_{b}\right]=1 ∀ 1\leq b\leq n-1.$

$$f\left[v\right]=f(v)\prod\_{b=1}^{n-1}f(v\_{b})$$

$$ =(-1)\prod\_{b=1}^{n-1}f(v\_{b})$$

 $=(-1)(1)$

 $=-1$

Hence $f$ is not a product signed dominating function.

Suppose for any odd $n,$ if the above pattern of assignment of functional values is followed, then $f\left[v\right]=1$ $⟺$ $\prod\_{b=1}^{n-1}f(v\_{b})=-1$ $ ⟺n-1=4k+1$

 $⟺n=4k+2$

 $⟺n≡2(mod4)$

but in this case, $f\left[v\_{1}\right]=f\left(v\_{1}\right)f\left(v\_{2}\right)f(v)f\left(u\_{1}\right)f\left(v\_{n-1}\right)$

 $=(-1)(-1)(-1)(-1)(-1)$

 $=-1$

Hence $f$ fails to be a product signed dominating function.

**Therefore, assigning** $-1$ **to** $v$ **under** $f$ **fails to give a product signed dominating function.**

**Let** $f\left(v\right)=1$**.**

Assign $f\left(v\_{1}\right)=1$. Then $f\left(u\_{1}\right)=1$.

Correspondingly, $f\left[v\_{1}\right]=f\left(v\right)f\left(v\_{1}\right)f\left(u\_{1}\right)f\left(v\_{2}\right)f(v\_{n-1})$

 $=(1)(1)(1)f\left(v\_{2}\right)f(v\_{n-1})$

 $=1$ if and only if $f\left(v\_{2}\right)$ and $f(v\_{n-1})$ are of same sign.

Suppose $f\left(v\_{2}\right)=f\left(v\_{n-1}\right)=1$. This procedure leads assigning $1$ to all the vertices of $G$ which gives a maximum weight.

So let us assign $f\left(v\_{2}\right)=f\left(v\_{n-1}\right)=-1$. Then $f\left(u\_{2}\right)=f\left(u\_{n-1}\right)=-1$.

Now, $f\left[v\_{2}\right]=f\left(v\right)f\left(v\_{1}\right)f\left(v\_{2}\right)f\left(v\_{3}\right)f(u\_{2}) $

 $=\left(1\right)\left(1\right)\left(-1\right)f(v\_{3})(-1)$

 $=1$ if and only if $f\left(v\_{3}\right)=1.$

Let $f\left(v\_{3}\right)=1$. Then $f\left(u\_{3}\right)=1.$

Now, $f\left[v\_{3}\right]=f\left(v\right)f\left(v\_{2}\right)f\left(v\_{3}\right)f\left(v\_{4}\right)f(u\_{3}) $

 $=\left(1\right)\left(-1\right)\left(1\right)f(v\_{4})(1)$

 $=1$ if and only if $f\left(v\_{4}\right)=-1.$

Repeating the above procedure, $f\left(v\_{5}\right)=1, f\left(v\_{6}\right)=-1, f\left(v\_{7}\right)=1, f\left(v\_{8}\right)=-1$ and so on. (i.e) $f(v\_{b})$ where $1\leq b\leq n-1$ follows the pattern $1,-1$ for every two vertices starting from $v\_{1}$ Therefore, if $n-1=2k,$ then the function is defined by $f\left(v\right)=1.$ $f\left(v\_{2k+1}\right)=1$ and $f\left(v\_{2(k+1)}\right)=-1$ for all $k=0 $to $\frac{n-3}{2}.$ Correspondingly, $f\left(u\_{2k+1}\right)=1$ and $f\left(u\_{2(k+1)}\right)=-1$ for all $k=0 $to $\frac{n-3}{2}$

Now by construction, $f\left[v\_{b}\right]=f\left[u\_{b}\right]=1 ∀ 1\leq b\leq n-1.$

$$f\left[v\right]=f(v)\prod\_{b=1}^{n-1}f(v\_{b})$$

 $=(1)(1)(-1)^{k}$

 $=1$ if and only if $k$ is even

 $=1$ if and only if $n-1$ is a multiple of $4$

Therefore, $f$ is a product signed dominating function when $n≡1\left(mod4\right).$

Now, $w\_{f}\left(G\right)=\sum\_{b=1}^{n-1}\left[f\left(u\_{b}\right)+f\left(v\_{b}\right)\right]+f(v)$

 $=0+f(v)$

 $=1$

Therefore, $ γ\_{sign}^{\*}\left(G\right)=\left\{\begin{array}{c}1 when n≡1(mod4)\\2n-1 otherwise\end{array}\right.$

**2.8 Illustration:**

$$-1$$

$$1$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$u\_{8}$$

$$u\_{7}$$

$$u\_{6}$$

$$1$$

$$u\_{5}$$

$$u\_{4}$$

$$u\_{3}$$

$$u\_{2}$$

$$u\_{1}$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$-1$$

$$1$$

$$v\_{2}$$

$$v\_{1}$$

$$v$$

$$v\_{8}$$

$$v\_{7}$$

$$v\_{6}$$

$$v\_{5}$$

$$v\_{4}$$

$$v\_{3}$$

**Figure 7**

**Product signed dominating function for graph** $G≅H\_{n}$ **on** $n=9≡1(mod 4)$ **vertices.**

$γ\_{sign}^{\*}\left(G\right)=1$.

**2.9 Theorem:**

$$γ\_{sign}^{\*}\left(Fl\_{n}\right)=\left\{\begin{array}{c}1 if n is odd \\2n-1 otherwise\end{array}\right.$$

**Proof:**

Let $Fl\_{n}$ represent a flower graph on $2n-1$ vertices.

Let $V=\left\{v,v\_{1},v\_{2},…,v\_{n-1},u\_{1},u\_{2},…,u\_{n-1}\right\}$ and $E=\left\{v\_{b}v\_{b+1}|1\leq b\leq n-2\right\}∪\left\{vv\_{b}|1\leq b\leq n-1\right\}∪\left\{1\leq b\leq n-1\right\}∪\left\{v\_{1}v\_{n-1}\right\}∪\{v\_{b}u\_{b}|1\leq b\leq n-1\}$

**Case 1:** $f\left(v\right)=-1$

Here to get any $f\left[u\_{b}\right] (1\leq b\leq n-1)$ as $1$, one of $f\left(u\_{b}\right), f(v\_{b})$ must be equal to $1$. But in this case, to get $f\left[v\_{b}\right]=1 ∀ b (1\leq b\leq n-1)$, $f$ should assign values to $u\_{b}$ and $v\_{b}$ for $1\leq b\leq n-1$ such that $\sum\_{b=1}^{n-1}f\left(u\_{b}\right)+\sum\_{b=1}^{n-1}f\left(v\_{b}\right)=0$. Finally, $w\_{f}\left(Fl\_{n}\right)=\sum\_{v\in V}^{}f\left(v\right)=-1$ which is negative.

Further to get $w\_{f}(Fl\_{n})$ as positive among the remaining $2n-2$ vertices atleast $n$ vertices must get $1$ under $f$.

But in this case, if one of $v\_{b}$ for $1\leq b\leq n-1$ gets $1$, then $f(v\_{b})=1 ∀ b (1\leq b\leq n-1)$ and $f\left(u\_{b}\right)=-1 ∀ b (1\leq b\leq n-1)$ so that $f\left[u\_{b}\right]=f\left[v\_{b}\right]=1 ∀ b (1\leq b\leq n-1)$.

**Subcase 1.1:** $n$ **is even**

Here $n-1$ is odd.

In this case $f$ is a valid product signed dominating function with $w\_{f}(Fl\_{n})$ negative.

**Subcase 1.2:** $n$ **is odd**

Then $n-1$ is even.

Here $f\left[v\right]=-1$ in which $f$ fails to be a product signed dominating function.

**Case 2:** $f\left(v\right)=1$

Here for every $b, 1\leq b\leq n-1$, both $u\_{b}$ and $v\_{b}$ must have the same functional value. That is, $f\left(u\_{b}\right)=f\left(v\_{b}\right)=1$ or $f\left(u\_{b}\right)=f\left(v\_{b}\right)=-1, 1\leq b\leq n-1.$

Suppose $f\left(u\_{k}\right)=f\left(v\_{k}\right)=1$ for some $k, 1\leq k\leq n-1$. Then the neighbor vertices of $v\_{k}$ in the inner cycle $(v\_{1}v\_{2}…v\_{n-1})$ must get $-1$ to get minimum weight. --- (I)

At the same time the neighbors of $v\_{k-1}$ and $v\_{k+1}$ must get $1$ (in the inner cycle) so that $f\left[v\_{k-1}\right]=f\left[v\_{k+1}\right]=1$.

Repeating this procedure, the vertices of the inner and outer cycle get $1$ and $-1$ alternately.

**Subcase 2.1:** $n$ **is even**

Therefore $n-1$ is odd.

Here the above procedure fails to give a valid product signed dominating function.

**Subcase 2.2:** $n$ **is odd**

Here $n-1$ is even.

In this case, the procedure yields a valid product signed dominating function and the corresponding

$$w\_{f}\left(Fl\_{n}\right)=f\left(v\right)+\sum\_{b=1}^{n-1}f(u\_{b})+\sum\_{b=1}^{n-1}f(v\_{b})$$

 $ =1+0$

 $=1$

Hence this $f$ is a product signed dominating function with a positive weight.

As the weight is $1$, this is minimum and the corresponding $γ\_{sign}^{\*}\left(Fl\_{n}\right)=1$.

Further by statement (I) and subcase 2.1, the only product signed dominating function giving positive weight is $f\left(v\right)=1 ∀ v\in V$ when $n-1$ is odd.

Hence $w\_{f}\left(Fl\_{n}\right)=\left|V\right|=2n-1$ when $n$ is even and the corresponding $γ\_{sign}^{\*}\left(Fl\_{n}\right)=2n-1$.

$$Therefore, γ\_{sign}^{\*}\left(Fl\_{n}\right)=\left\{\begin{array}{c}1 if n is odd \\2n-1 otherwise\end{array}\right.$$

**2.10 Illustration:**

$$1$$

$$v$$

$$v\_{1}$$

$$u\_{1}$$

$$u\_{2}$$

$$v\_{2}$$

$$v\_{3}$$

$$u\_{3}$$

$$v\_{4}$$

$$u\_{4}$$

$$1$$

$$1$$

$$-1$$

$$-1$$

$$1$$

$$1$$

$$-1$$

$$-1$$

**Figure 8**

 **Product signed dominating function for flower graph** $Fl\_{5}$ **on** $2\left(5\right)-1=9$ **vertices.**

$γ\_{sign}^{\*}\left(Fl\_{5}\right)=$ $1$.

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