1.1 INTRODUCTION

This chapter deals with the Fibonacci and Lucas sequences [5,6]. To consider all the sequences [7,8] of this type under one heading we are introducing generalized Fibonacci sequence $\{G_n\}$, which is defined by [3] –

$$G_n = G_{n-1} + G_{n-2},\ \ n \geq 3 \qquad (1.1.1)$$

$$G_1 = a\ ,\ \ G_2 = b$$

where, a and b are constants.

But this generalization does not include Pell sequence, associated Pell sequence, and other sequences which can be similarly defined. To discuss all such sequences under one heading we consider the sequence $\{S_n\}$, defined by –

$$S_{n+1} = k\, S_n + S_{n-1} \qquad (1.1.2)$$

where, S_o and S_1 are arbitrary constants and k is any number.

We shall also introduce the two companion sequences $\{H_n\}$ and $\{V_n\}$ defined by [4] –

$$H_{n+1} = k\, H_n + H_{n-1}\ (n \geq 1),\ H_o = 0,\ H_1 = 1 \qquad (1.1.3)$$

$$V_{n+1} = k\, V_n + V_{n-1}\ (n \geq 1),\ V_o = 2,\ V_1 = k \qquad (1.1.4)$$

Thus the sequence $\{H_n\}$ and $\{V_n\}$ are particular cases of the sequences $\{S_n\}$. The sequence $\{H_n\}$ reduces to Fibonacci or Pell sequence according as k = 1 or 2. Similarly the sequence $\{V_n\}$ reduce to Lucas or associated Pell sequence according as k = 1 or 2.

1.2 DIFFERENCE OPERATORS E AND Δ

In this section we shall derive some new identities for the generalized Fibonacci numbers G_n, with the help of finite difference operators E and Δ. These operators are defined by [1] –

$$E\, G_n = G_{n+1} \tag{1.2.1}$$

$$\Delta\, G_n = G_{n+1} - G_n, \text{ and} \tag{1.2.2}$$

$$\Delta \equiv E - 1 \tag{1.2.3}$$

Also, it is known that,

$$\Delta\, (A_n\, B_m) = (E\, A_n)\, (\Delta\, B_m) + B_m\, (\Delta\, A_n) \tag{1.2.4}$$

Since,

$$G_{n+1} - G_n = G_{n-1}$$

Therefore, we obtain

$$\Delta\, G_n = G_{n-1}$$

and in general,

$$\Delta^m\, G_n = G_{n-m} \tag{1.2.5}$$

Using the operators we obtain the following identities :

$$G_{n-m} = \sum_{p=0}^{m} (-1)^p \binom{m}{p} G_{n+m-p} \tag{1.2.6}$$

$$G_{n+m} = \sum_{p=0}^{m} \binom{m}{p} G_{n-p} \tag{1.2.7}$$

$$\sum_{p=0}^{m} \binom{m}{p} G_{n+p} = \sum_{p=0}^{m} \binom{m}{p} 2^{m-p} G_{n-p} \tag{1.2.8}$$

$$G^2_{n+1} - G_n^2 = G_{n+1} G_{n-1} + G_n G_{n-1} \tag{1.2.9}$$

Proof

(1) From equations (1.2.5), (1.2.3) and (1.2.1), we obtain

$$\begin{aligned} G_{n-m} &= \Delta^m G_n \\ &= (E - 1)^m G_n \\ &= \sum_{p=0}^{m} (-1)^p G_{n+m-p} \end{aligned}$$

which is the result of eqn. (1.2.6).

(2) We have

$$\begin{aligned} G_{n+m} &= E^m G_n \\ &= (1 + \Delta)^m G_n \\ &= \sum_{p=0}^{m} \binom{m}{p} G_{n-p} \end{aligned}$$

which is the result of eqn. (1.2.7).

(3) We have,

$$(1 + E)^m G_n = \sum_{p=0}^{m} \binom{m}{p} G_{n+p}$$

Now,

$$\begin{aligned} \text{L.H.S.} &= (2 + \Delta)^m G_n \\ &= \sum_{p=0}^{m} \binom{m}{p} 2^{m-p} G_{n-p} \end{aligned}$$

which is the result of eqn. (1.2.8).

(4) Setting $A_n = B_m = G_n$ in eqn. (5.2.4), we get

$$\Delta (G_n G_n) = (E G_n)(\Delta G_n) + G_n (\Delta G_n)$$

which leads to the result of eqn. (1.2.9).

1.3 EXPLICIT EXPRESSIONS FOR S_n, H_n and V_n

An explicit expression for the numbers S_n, may be derived by the usual method for solving difference equations. By this method we deduce that [2] –

$$S_n = X \alpha^n + Y \beta^n \qquad (1.3.1)$$

where, α and β are the roots of the equation.

$$t^2 - kt - 1 = 0 \qquad (1.3.2)$$

and X and Y are suitably chosen constants depending on S_0 and S_1.

From eqn. (1.3.2) we get,

$$\alpha = ½ [k + (k^2 + 4)^{1/2}]$$

$$\beta = ½ [k - (k^2 + 4)^{1/2}]$$

giving,

$$\left.\begin{aligned} &\alpha + \beta = k, \\ &\alpha \beta = -1, \\ &\alpha - \beta = (k^2 + 4)^{1/2} \end{aligned}\right\} \qquad (1.3.3)$$

It follows that –

$$X = \frac{S_1 - S_0\beta}{\alpha - \beta}$$

$$Y = \frac{S_1 - S_0\alpha}{\alpha - \beta} \tag{5.3.4}$$

From eqns. (1.3.1) and (1.3.4), we obtain

$$S_n = \frac{(\alpha^n - \beta^n)S_1 + (\alpha^{n-1} - \beta^{n-1})S_0}{\alpha - \beta} \tag{1.3.5}$$

Taking $S_0 = 0$ and $S_1 = 1$, the sequence S_n reduces to H_n, and from eqn. (1.3.5) we obtain,

$$H_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.3.6}$$

Taking, $S_0 = 2$ and $S_1 = k$, the sequence $\{S_n\}$ is reduces to $\{V_n\}$ and using eqn. (5.3.3), we obtain,

$$V_n = \alpha^n + \beta^n \tag{1.3.7}$$

From eqns. (1.3.5) and (1.3.6), we have

$$S_n = H_n S_1 + H_{n-1} S_0 \tag{1.3.8}$$

Now we shall derive some identities involving these numbers.

Since α is a root of the equation (1.3.2), we have

$$\alpha^2 = 1 + k\,\alpha \tag{1.3.9}$$

therefore,

$$\alpha^{2n} = \sum_{p=0}^{n} \binom{n}{p} k^p \alpha^p \tag{1.3.10}$$

Similarly, we get

$$\beta^{2n} = \sum_{p=0}^{n} \binom{n}{p} k^p \alpha^p \tag{1.3.11}$$

Hence, from eqns. (1.3.1), (1.3.10) and (1.3.11), we obtain the identity –

$$S_{2n} = \sum_{p=0}^{n} \binom{n}{p} k^p S_p \qquad (1.3.12)$$

Denoting ($S_0^2 + k\, S_0\, S_1 - S_1^2$) by δ, and using eqns. (1.3.1), (1.3.3) and (5.3.6), the following identities have been obtained :

$$S_n^2 - S_{n-1}\, S_{n+1} = (-1)^n\, \delta \qquad (1.3.13)$$

$$S_m\, S_{n+p} - S_{m+p}\, S_n = (-1)^m\, \delta\, H_p\, H_{n-m} \qquad (1.3.14)$$

$$S_{m+n+1} = S_{m+1}\, H_{n+1} + S_m\, H_n \qquad (1.3.15)$$

$$S_{m+r}\, H_{n+r} + (-1)^{r+1}\, S_m\, H_n = S_{m+n+r}\, H_r \qquad (1.3.16)$$

$$H_n\, H_{n+m} - H_{n-s}\, H_{n+m+s} = (-1)^{n-s}\, H_s\, H_{s+m} \qquad (1.3.17)$$

$$H_{-n} = (-1)^{n-1}\, H_n \qquad (1.3.18)$$

$$S^2_{n+1} - S^2_{n-1} = S_2\, S_{2n} - S_0\, S_{2n-2} \qquad (1.3.19)$$

$$H_n\, V_n = H_{2n} \qquad (1.3.20)$$

Similar results can be obtained for other sequences by suitably choosing the constants.

1.4 GEOMETRICAL PROPERTIES OF $\{S_n\}$ AND $\{H_n\}$

Theorem 1

Area of the triangle having vertices at the points with rectangular Cartesian coordinates (S_n, S_{n+r}), (S_{n+a}, S_{n+a+r}), (S_{n+b}, S_{n+b+r}) is independent of n.

Proof

Twice the area of the specified triangle is equal to the absolute value of the determinant.

$$\Delta = \begin{vmatrix} S_n & S_{n+r} & 1 \\ S_{n+a} & S_{n+a+r} & 1 \\ S_{n+b} & S_{n+b+r} & 1 \end{vmatrix}$$

Using (1.3.15) for the second column the determinant can be written as –

$$H_{r+1} \begin{vmatrix} S_n & S_{n+r} & 1 \\ S_{n+a} & S_{n+a} & 1 \\ S_{n+b} & S_{n+b} & 1 \end{vmatrix} + H_r \begin{vmatrix} S_n & S_{n+r} & 1 \\ S_{n+a} & S_{n+a-1} & 1 \\ S_{n+b} & S_{n+b-1} & 1 \end{vmatrix}$$

The first determinant is obviously zero; in the second on alternately subtracting the second and first columns from each other, the suffixes can be reduced and finally we get –

$$\Delta = \underline{+}\, H_r \begin{vmatrix} S_1 & S_0 & 1 \\ S_{a+1} & S_a & 1 \\ S_{b+1} & S_b & 1 \end{vmatrix},$$

according as n is odd or even.

On expanding the determinant along the third column we obtain,

$$\Delta = \underline{+}\, H_r \,[\,(S_{a+1}\, S_b – S_a\, S_{b+1}) – (S_1\, S_b – S_0\, S_{b+1})$$

$$+ (S_1\, S_a – S_0\, S_{a+1}),$$

which on using (1.3.14), reduces to –

$$\Delta = \underline{+}\, H_r\, [H_b – H_a – (-1)^a\, H_{b-a}]\, \delta \qquad (1.4.1)$$

Thus the area of the specified triangle is independent of n.

Particular Case

Taking $T_o = 0$, $T_1 = 1$, $k = 1$ and $e = h$, $a = 2h$, $b = 4h$, we find that the area of the triangle whose vertices are (F_n, F_{n+h}), (F_{n+2h}, F_{n+3h}), (F_{n+4h}, F_{n+5h}) is equal to –

$$\tfrac{1}{2} F_h (F_{rh} - 2F_{2h}) \qquad (1.4.2)$$

Duncan has proved that the area of this triangle is,

$$\tfrac{1}{2} [F_h(F_{4h} - F_{2h}) - (F_{3h} F_{4h} - F_{2h} F_{5h})] \quad ,$$

which on using (5.3.17) simplifies to the expression given in (1.4.2).

Theorem 2

Lines drawn through the origin with direction ratios T_n, T_{n+a}, T_{n+b}, where, a and b are any integers are always co-planar for every value of n.

Proof

Direction ratios of any three such lines are S_e, S_{e+a}, S_{e+b}, S_f, S_{f+a}, S_{f+b}, S_g, S_{g+a}, S_{g+b}. these will be coplanar if

$$\begin{vmatrix} S_e & S_{e+a} & S_{e+b} \\ S_f & S_{f+a} & S_{f+b} \\ S_g & S_{g+a} & S_{g+b} \end{vmatrix} = 0 \qquad (1.4.3)$$

On using the relation (1.3.15) the left-hand side of (1.4.3) can be written as the sum of four determinants each of which is zero. Hence, the theorem is proved.

Theorem 3

The points having Cartesian coordinates (S_n, S_{n+a}, S_{n+b}), where, a and b are any integers and n = 1,2,3,….., are always co-planar and the plane through these points passes through the origin, and its equation is independent of n.

Proof

Equation to the plane passing through any three such points is –

$$\begin{vmatrix} x & y & z & 1 \\ S_e & S_{e+a} & S_{e+b} & 1 \\ S_f & S_{f+a} & S_{f+b} & 1 \\ S_g & S_{g+a} & S_{g+b} & 1 \end{vmatrix} = 0 \qquad (1.4.4)$$

where, e, f and g are particular values of n.

Here the coefficient of x is

$$= [(S_{f+a} S_{g+b} – S_{f+b}S_{g+a}) – (S_{e+a} S_{g+b} – S_{e+b} S_{g+a}) + (S_{e+a} S_{f+b} – S_{e+b} S_{g+a})]$$

$$= (-1)^a [H_{b-a}(-1)^f H_{g-f} – (-1)^e H_{g-e} + (-1)^e H_{f-e}) \delta$$

The coefficient of y is obtained on putting a = 0 in the coefficient of x; the coefficient of z is obtained from the coefficient of y on replacing b by a; the constant term is zero as is already proved in (1.4.3).

Thus the equation to the plane simplifies to -

$$(-1)^a H_{b-a} x – H_b y + H_a z = 0 \qquad (1.4.5)$$

This equation is independent of n. Also it does not depend on the initial values T_o and T_1.

Particular Case

From (1.4.5) we obtain that the points (F_e, F_{e+2}, F_{e+5}), e = 1,2,3,…; (L_f, L_{f+2}, L_{f+5}), f = 1,2,3,…; (G_g, G_{g+2}, G_{g+5}), g = 1,2,3,…; all lie on the plane $2x - 5y + z = 0$.

Theorem 4

The planes –

$S_n x + S_{n+a} y + S_{n+b} z + S_{n+r} = 0,$

where, a, b, r are any integers and n = 1,2,3,…; all intersect in a given line whose equation is independent of n.

Proof

Let two such planes be –

$$S_e\, x + S_{e+a}\, y + T_{e+b}\, z + S_{e+r} = 0, \qquad (1.4.6)$$

$$S_f\, x + S_{f+a}\, y + S_{f+b}\, z + S_{f+r} = 0.$$

The equations to the line of intersection of the parallel planes through the origin are

$$\frac{x}{S_{e+a}S_{f+b} - S_{e+b}S_{f+a}} = \frac{y}{S_e S_{f+b} - S_{e+b}S_f} = \frac{z}{S_e S_{f+b} - S_{e+b}S_f}$$

On using (5.3.14) and preceding as in (5.4.5) we obtain the equation of the line of intersection of the parallel planes through the origin as –

$$\frac{x}{(-1)^a H_{b-a}} = \frac{-y}{H_b} = \frac{z}{H_a}$$

The line of intersection of the planes given by (5.4.6) meets the plane z = 0, at the point given by –

$$\frac{x}{(-1)^a H_{r-a}} = \frac{-y}{H_r} = \frac{1}{H_a}$$

Thus the equation to the line of intersection of the planes given by (1.4.6) becomes

$$\frac{(-1)^p H_a x - H_{r-a}}{H_{b-a}} = \frac{H_a y + H_r}{-H_b} = \frac{z}{H_a}, \qquad (1.4.7)$$

which is independent of n.

Particular case

The planes whose equations are –

$F_e\, x + F_{e+1}\, y + F_{e+3}\, z + F_{e+4} = 0$, e = 1,2,3,….,

$L_f\, x + L_{f+1}\, y + L_{f+3}\, z + L_{f+4} = 0$, f = 1,2,3,….,

$B_g\, x + B_{g+1}\, y + B_{g+3}\, z + B_{g+4} = 0$, g = 1,2,3,….,

all intersect in the line $\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{-1}$

1.5 A THIRD-ORDER DETERMINANT INVOLVING THE NUMBERS S_n

From (1.4.3) it follows that

$$\begin{vmatrix} S_a & S_{a+m} & S_{a+m+n} \\ S_b & S_{b+m} & S_{b+m+n} \\ S_r & S_{r+m} & S_{r+m+n} \end{vmatrix} = 0 \qquad (1.5.1)$$

for all integers a, b, r, m and n.

We shall now evaluate the determinant,

$$\Delta_1 = \begin{vmatrix} S_a + e & S_{a+m} + e & S_{a+m+n} + e \\ S_b + e & S_{b+m} + e & S_{b+m+n} + e \\ S_r + e & S_{r+m} + e & S_{r+m+n} + e \end{vmatrix}$$

where, e is an arbitrary constant and a, b, r, m and n are any integers.

On writing the determinant as the sum of eight determinants; using the equation (1.5.1) and the property that a determinant vanishes if two columns are identical, we obtain,

$$\Delta_1 = \begin{vmatrix} S_a & S_{a+m} & e \\ S_b & S_{b+m} & e \\ S_r & S_{r+m} & e \end{vmatrix} + \begin{vmatrix} \ldots \\ \ldots \\ \ldots \end{vmatrix} + \begin{vmatrix} \ldots \\ \ldots \\ \ldots \end{vmatrix}$$

$$= e H_m \begin{vmatrix} S_a & S_{a-1} & 1 \\ S_b & S_{b-1} & 1 \\ S_r & S_{r-1} & 1 \end{vmatrix} + \ldots + \ldots \ .$$

The first determinant by using (1.3.14) can be written as –

$$= \delta e H_m \left[(-1)^{r-1} H_{b-r} + (-1)^{a-1} H_{r-a} + (-1)^{b-1} H_{-b+a}\right]$$

Hence,

$$\Delta_1 = \delta g \, [(-1)^b H_{r-b} - (-1)^a H_{r-p} + (-1)^a H_{b-a}] \, [H_m - H_{m+n} + (-1)^m H_n] \quad (5.5.2)$$

1.6 FOURTH-ORDER DETERMINANTS

We shall now evaluate the determinant,

$$\Delta_2 = \begin{vmatrix} S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+2} & S_{n+3} & S_n & S_{n+1} \\ S_{n+1} & S_n & S_{n+3} & S_{n+2} \\ S_n & S_{n+1} & S_{n+2} & S_{n+3} \end{vmatrix},$$

Hence, we obtain,

$$\Delta_2 = [(S_{n+3} + S_{n+2})^2 - (S_{n+1}+S_n)^2] \, [(S_{n+3} - S_{n+2})^2 - (S_{n+1} - S_n)^2]$$

$$= (S^2_{n+4} - S^2_{n+2})(S^2_{n+1} - S^2_{n-1})$$

$$= (S_2 S_{2n+6} - S_0 S_{2n+4}) \, (S_2 S_{2n} - S_0 S_{2n-2}) \quad (1.6.1)$$

on using (1.3.19).

1.7 EVALUATION OF A CIRCULANT

We now evaluate the circulant

$$\Delta_3 = \begin{vmatrix} S_n & S_{n+g} & \cdots & S_{n+(m-1)g} \\ S_{n+(m-1)g} & S_n & \cdots & S_{n+(m-2)g} \\ \cdots & \cdots & \cdots & \cdots \\ S_{n+g} & S_{n+2g} & \cdots & S_n \end{vmatrix}$$

Let w be any one of the m numbers,

$$w_r = \cos\frac{2r\pi}{m} + i\sin\frac{2r\pi}{m}, \ (r = 1,2,3,\ldots, m)$$

so that,

$$w^m - 1 = 0$$

Therefore, $T_1 = w_1 + w_2 + w_2 + \ldots + w_m = 0$

$T_2 = w_1 . w_2 + \ldots = 0$

$\ldots \quad \ldots \quad \ldots = 0$

$T_m = w_1 w_2 w_3 w_4 \ldots w_m = (-1)^{m+1}$

Hence,

$$\prod_{r=1}^{m}(y - w_r z) = y^m - z^m \tag{1.7.1}$$

Therefore, as discussed in -

$$\Delta_3 = \prod_{r=1}^{m}\left(S_n + w_r S_{n+k} + \ldots + w_r^{m-1} S_{n+(m-1)g}\right)$$

$$= \prod_{r=1}^{m}\left[\frac{C\alpha^{n}(1-w_r^{m}\alpha^{mg})}{1-w_r\alpha^{g}}+\frac{D\beta^{n}(1-w_r^{m}\beta^{mg})}{1-w_r\beta^{g}}\right]$$

$$= \prod_{r=1}^{m}\left[\frac{(T_n - T_{n+mg}) - (-1)^{g} w_r (S_{n-k} - S_{n+(m-1)g})}{(1-w_r\alpha^{g})(1-w_r\beta^{g})}\right]$$

$$= \frac{(S_n - S_{n+mk})^{m} - (-1)^{mk}(S_{n-g} - S_{n+(m-1)g})^{m}}{(1-\alpha^{mg})(1-\beta^{mg})}$$

$$= \frac{(S_n - S_{n+mk})^{m} - (-1)^{mk}(S_{n-g} - S_{n+(m-1)g})^{m}}{1+(-1)^{mg} - w_{mg}} \qquad (1.7.2)$$

1.8 A THIRD-ORDER DETERMINANT WITH EACH ELEMENT AS THE PRODUCTS OF TWO NUMBERS

We shall evaluate,

$$\Delta_4 = \begin{vmatrix} H_n.S_{m+n} & H_{n+a}.S_{m+n+a} & H_{n+a+b}.S_{m+n+a+b} \\ H_{n+r}.S_{m+n+r} & H_{n+r+a}.S_{m+n+r+a} & H_{n+r+a+b}.S_{m+n+r+a+b} \\ H_{n+s}.S_{m+n+s} & H_{n+s+a}.S_{m+n+s+a} & H_{n+s+a+b}.S_{m+n+s+a+b} \end{vmatrix}$$

and shall show that Δ_4 is independent of n.

On using (1.3.16), we can write,

$H_{n+a}\ S_{m+n+a} + (-1)^{a+1}\ H_n\ S_{m+n} = H_a\ S_{m+2n+p}$

Hence, multiplying the first column by $(-1)^{a+1}$, $(-1)^{a+b+1}$ and adding to the second and third columns respectively, we obtain,

$$\Delta_4 = H_aH_{a+b}\begin{vmatrix} H_n.S_{m+n} & S_{m+2n+a} & S_{m+2n+a+b} \\ H_{n+r}.S_{m+n+r} & S_{m+2n+2r+a} & S_{m+2n+2r+a+b} \\ H_{n+s}.S_{m+n+s} & S_{m+2n+2s+b} & S_{m+2n+2s+a+b} \end{vmatrix}$$

$$\Delta_4 = H_a H_{a+b} \begin{vmatrix} H_n.S_{m+n} & S_{m+2n+a} & S_{m+2n+a-1} \\ H_{n+r}.S_{m+n+r} & S_{m+2n+2r+a} & S_{m+2n+2r+a-1} \\ H_{n+s}.S_{m+n+s} & S_{m+2n+2s+a} & S_{m+2n+2s+a-1} \end{vmatrix}$$

on using (1.3.15),

Now alternately subtracting the third and second columns from one another, we obtain,

$$\Delta_4 = H_a\, H_b\, H_{a+b}\, (-1)^{m+a} \begin{vmatrix} H_n.S_{m+n} & S_0 & S_1 \\ H_{n+r}.S_{m+n+r} & S_{2r} & S_{2r+1} \\ H_{n+s}.S_{m+n+s} & S_{2s} & S_{2s+1} \end{vmatrix}$$

$$= H_a\, H_b\, H_{a+b}\, (-1)^{m+a}\, [H_n\, S_{m+n}\, H_{2s-2r} - H_{n+r}\, S_{m+n+r}\, H_{2s} +$$

$$+ H_{n+s}\, S_{m+n+s}\, H_{2r}\,]\, \delta$$

on using (1.3.14)

Now with the help of equations (1.3.5), (1.3.6) and (1.3.7) we can write,

$$H_{n+s}\, S_{m+n+s}\, H_{2r} = [S_{m+2n+2s+2r} - S_{m+2n+2s-2r} + (-1)^{n+s}\, (S_{m-2r} - S_{m+2r})]\, /\, (\alpha-\beta)^2$$

Hence, we get,

$$\Delta_4 = \frac{(-1)^{m+n+a} H_a H_b H_{a+b}}{w_2 + 2} [(S_{m+2r-2s} - S_{m+2s-2r}) +$$

$$+\ (-1)^s\, (S_{m-2r} - S_{m+2r}) - (-1)^r\, (S_{m-2s} - S_{m+2s})\,]\, \delta \quad (5.8.1)$$

which is independent of n.

1.9 A FOURTH-ORDER DETERMINANT WITH ELEMENTS AS PRODUCTS OF TWO NUMBERS

We shall now show that,

$$\Delta_5=\begin{vmatrix} H_aS_{a+m} & H_{a+o}S_{a+m+o} & H_{a+p}S_{a+m+p} & H_{a+q}S_{a+m+q} \\ H_bS_{b+m} & H_{b+o}S_{b+m+o} & H_{b+p}S_{b+m+p} & H_{b+q}S_{b+m+q} \\ H_rS_{r+m} & H_{r+o}S_{r+m+o} & H_{r+p}S_{r+m+p} & H_{r+q}S_{r+m+q} \\ H_sS_{s+m} & H_{s+o}S_{s+m+o} & H_{s+p}S_{s+m+p} & H_{s+q}S_{s+m+q} \end{vmatrix} = 0 \quad (1.9.1)$$

For all integers p, q, r, s, m, o, p and q.

Multiplying the first column by $(-1)^{a+1}$, $(-1)^{b+1}$, $(-1)^{c+1}$ and adding to the second, third and fourth columns respectively; using equation (1.3.16) the determinant reduces to

$$H_o\,H_p\,H_q = \begin{vmatrix} H_aS_{a+m} & S_{2a+m+o} & S_{2a+m+p} & S_{2a+m+q} \\ H_bS_{b+m} & S_{2b+m+o} & S_{2b+m+p} & S_{2b+m+q} \\ H_rS_{r+m} & S_{2r+m+o} & S_{2r+m+p} & S_{2r+m+q} \\ H_sS_{s+m} & S_{2s+m+o} & S_{2s+m+p} & S_{2s+m+q} \end{vmatrix}$$

Expanding along the first column and using the equation (1.5.1), we find that the determinant vanishes. The result (1.9.1) can be extended for the n^{th} order determinants.

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