**Radial Radio Number and some other labeling parameters**

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**ABSTRACT**

Let G(V(G), E(G)) be a simple, connected, and undirected graph. A radial radio labeling of G is an assignment of positive integers to the vertices, such that for any two distinct vertices u, v ∈ V(G), the inequality d(u, v) + | (u) - (v)| ≥ 1 + rad(G) holds, where d(u, v) represents the distance between vertices u and v, and rad(G) is the radius of the graph G. The span of a radial radio labeling is defined as the highest integer value in the range of and is denoted as span . In this paper, we establish the relationships among the radial radio number, the radio number, and the L(2,1)-labeling number. Furthermore, we construct specific graphs where the radio number equals the algebraic sum of the radial radio number and a given nonnegative integer. Similarly, we provide a proof of the existence of graphs for which the L(2,1)-labeling number is the algebraic sum of the radial radio number and a given nonnegative integer.

**Keywords –** radial radio number, radio number, labeling number.

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**I. INTRODUCTION**

In this paper, we only consider a simple, connected, finite and undirected graph. The radius of G is denoted by rad(G) or r and the diameter of G is denoted by diam(G) or d.For further details, one can refer [3].

Graph labeling is an assignment of nonnegative integers, sometimes called colors, to the vertices or edges or both. Motivated by the Frequency Assignment Problem[7], numerous mathematicians introduced various graph labeling concepts. Some of them are discussed in this paper, namely, L(2,1) – labeling[6], radio labeling[4] and radial radio labeling[8].

The concept of **L(2,1)-labeling** was introduced by Griggs and Yeh[6]. It is defined as a function : V(G) → {1, 2, 3, ...} that adheres to the following conditions for any two distinct vertices u and v in graph G:

(i) | (u) - (v)| ≥ 2 if the distance d(u, v) = 1

(ii) | (u) - (v)| ≥ 1 if d(u, v) equals 2.

A **k-L(2,1) labeling** is an L(2,1)-labeling with the additional constraint that no label exceeds the value of k. The **L(2,1)-labeling number** of G is denoted by λ(G) and represents the smallest integer value k for which G possesses a k-L(2,1) labeling.

The notion of radio labeling was originally introduced by Chartrand et al[4]. A function : V(G) → N is considered a **radio labeling** if it adheres to the condition:

 (\*)

for any distinct vertices u and v in graph G. This condition is referred to as the radio condition. The span of a radio labeling is the largest integer in the range of and is denoted as span(). The **radio number**, denoted as rn(G), is defined as the minimum span taken over all possible radio labelings of G.

Motivated by the frequency assignment problem[7] and the concept of radio labeling[4], KM. Kathiresan and S. Vimalajenifer introduced a novel graph labeling known as radial radio labeling. A **radial radio labeling**, , is an assignment of positive integers to all vertices in such a way that it satisfies the condition:

 (\*\*)

for any distinct vertices u and v in G. The span of a radial radio labeling, , is the largest integer in the range of and is denoted as span . The **radial radio number**, rr(G), is defined as the minimum span taken over all possible radial radio labelings of G. Mathematically, this can be expressed as:

Below are listed a few fundamental outcomes that aid in the subsequent advancement of this paper:

**Theorem A:** , .[4]

**Theorem B:** , .[6]

**Theorem C:** For any self – centered graph G, .[1]

**Theorem D:** If , then .[1]

**Theorem E:** For any simple connected graph G, , where is the clique number.[1]

**Theorem F:** For any simple connected graph , and where is the maximum degree in G.[1]

 This paper focuses on establishing the correlations among the radial radio number, radio number, and the L(2,1) – labeling number. Moreover, we substantiate the existence of graphs where the radio number is the algebraic sum of its radial radio number and any non-negative integer. Furthermore, we construct graphs wherein the L(2,1) – labeling number is the algebraic sum of its radial radio number and any non-negative integer.

**II. RELATIONS CONNECTING RADIAL RADIO NUMBER, RADIO NUMBER AND L(2,1) – LABELING NUMBER**

 Throughout this section, we solely contemplate simple connected graphs on n vertices. The first two theorems provide the relationship between and .

**Theorem 2.1**

 **If radius of G is 1, then .**

**Proof**

 Assume that and is one of the radial radio labelings of G such that . Then by definition, for any two distinct vertices u and v of G, satisfies:

 (3)

Inequality (3) implies that,

1. if , then
2. if , then

From i) and ii), we observe that does not satisfy the L(2,1) – labeling condition so that the set of positive integers is not sufficient to label the vertices of G under – labeling condition. This forces that, .

**Theorem 2.2**

 **If , then .**

**Proof**

 Let be a radial radio labeling of G such that and let . By (\*\*) satisfies . If u and v are adjacent, then and if u and v are non adjacent, then . This implies that, .

**Theorem 2.3**

**If diameter of G is 1, then .**

**Proof**

If diam(G)=1, then G must be isomorphic to the complete graph . By Theorems A and B, it is clear that .

From this theorem, we deduce that:

**Corollary 2.4**

 **If diameter of G is 1, then .**

**Proof**

 If diam(G)=1, then G is self – centered with radius 1 and so [D]. By Theorem 2.1, .

**Theorem 2.5**

 **If , then .**

**Proof**

Assume that, and are and radio labeling of G, respectively, such that and . Let . Then satisfies the radio condition

 (4)

**Case 1:** whendiam(G)=2

1. if , then (4) becomes and
2. if , then (4) becomes

Here the statements in i) and ii) are as same as the – labeling conditions and hence . Thus, in this case, .

**Case 2:** when diam(G)>2

1. If , then (4) becomes and
2. if , then (4) becomes

In this case also, satisfies the – labeling conditions. But the strict inequalities in iii) and iv) forces that, . Thus .

This completes the proof.

**Corollary 2.6**

If G is self – centered with , then .

Overall, from this section, we can assert that:

**Theorem 2.7**

For any simple connected graph G,

1. if and , then .
2. if and , then .
3. if G is self – centered with , then .
4. if G is not self – centered and , then

**III. RADIAL RADIO NUMBER AND RADIO NUMBER**

Within this section, we demonstrate the existence of graphs in which the radio number equals the algebraic sum of the radial radio number and a specified nonnegative integer.

**Theorem 3.1**

**There is a graph satisfying the condition that , where m=0.**

**Proof**

 Let us take . Since , .

**Theorem 3.2**

 **There is no graph exists, for which .**

**Proof**

Since for each self – centered graph , .[1] Assume that, is not self – centered. That is, , which implies that, . As per Theorem F, we have and so

 (5)

Also, by Theorem F, we have

 (6)

 implies that, , and so , since G is connected. Thus, there exists no graph such that, .

**Theorem 3.3**

 **There exists a graph G, for which , where .**

**Proof**

Assume that G is constructed by using two copies of , . Let and let . Then and .

We now find the radial radio number for G.

Define such that , where . We have to prove that, for every pair of vertices u and v of G, satisfies,

 . (7)

**Case 1a:** Consider the pair

Here, for all . Since, the pair satisfies (7). In a similar manner, we can show that the pair also satisfies (7), for all .

**Case 2a:** Consider the pair

Since and , the pair satisfies (7) for all , . Similarly, we can prove that the pair satisfies (3) for all , .

**Case 3a:** Consider the pair .

In this case, . We have, . Thus pair also satisfies (7).

 From the three cases, we can say that, is a radial radio labeling of G and , which implies that Also, , by Theorem E, . Thus .

Now, we determine the radio number for G. Define such that , ; , . We have to prove that, for every pair of vertices u and v of G, satisfies,

 . (8)

**Case 1b:** Consider the pair , .

Since , for all , , and hence the pair , satisfies (8), for all . Proceeding like this, we can show that the pair , satisfies (8), for all ,

**Case 2b:** Consider the pair

Here Also, , the pair satisfies (4) for all , . Similarly, we can prove that the pair satisfies (8) for all , .

**Case 3b:** Consider the pair .

We have, and . Thus pair also satisfies (8).

 From cases 1b, 2b, 3b, we arrive at a conclusion that, is a radio labeling for . Also, we have , which forces that, . Since , by Theorem E, . Thus . Finally, for this graph and , which implies that, . For m=5 the graph G is illustrated in Figure 1. The corresponding radio labeling and radial radio labeling are illustrated in Figure 2 and Figure 3, respectively.

**Figure 1**

**Figure 2 Figure 3**

From Theorems 3.1, 3.2 and 3.3, we note that:

**Theorem 3.4**

**There is a graph G such that , where , and m is given nonnegative integer.**

**IV. RADIAL RADIO NUMBER AND L(2,1) – LABELING NUMBER**

 In this section, we show the existence of graphs for which the L(2,1) – labeling number is the algebraic sum of the radial radio number and any given non negative integer.

**Theorem 4.1**

 **For , there exists graph G such that .**

**Proof**

**Case 1:** m=0

Take , where . We have . In this case, is the required graph.

**Case 2:** m=1

Consider G as . We know that, and .

**Case 3:** m=2

In this case, is the desired graph, since and .

**Theorem 4.2**

**For , there exists graph G such that .**

**Proof**

Let and . Then rad(G)=1 and diam(G)=2.

Define such that , ; ; ; ; , . We now show that, is an L(2,1) – labeling of G.

1. for the pair , ,
2. for the pair , ,
3. for the pair , ,
4. for the pair , , ,

Form this discussion, if two vertices u and v are adjacent, then the label difference between them is atleast 2 and if two vertices u and v are non adjacent, then the label difference between them is atleast 1 and so is an L(2,1) – labeling. Also, . We know that, . Since , . Thus we conclude that, .

Now, we determine the radial radio number of G. Define such that , and , . Here, we note the following:

 , ;

 , and ;

 , ;

 , .

This implies that, every pair of vertices of G satisfies (\*\*) that and hence is a radial radio labeling of G and and hence . Since , by Theorem E, . This gives that, . Hence G is the required graph.

For m=7, the constructed graph is drawn in Figure 4. The corresponding L(2,1) – labeling and radial radio labeling are shown in Figure 5 and Figure 6, respectively.

**Figure 4**

**Figure 5 Figure 6**

**V. CONCLUSION**

In this paper, we compared three labeling parameters, which are based on the distance between two vertices of a graph G. Also, we prove the existence of graphs whose radio number and L(2, 1) – labeling number as the algebraic sum of radial radio number and any given non negative integer. In a similar manner, we may compare other graph labeling parameters.

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