**Local Isolate Domination in Graphs**

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**Abstract**

A dominating set S of G is said to be an isolate dominating set if

*< S >* has at least one isolated vertex. A new variant of domination called local isolate dominating set which is defined as the set S such that for each *u∈S, < N* (*u*) *>* has an isolated vertex is introduced. The minimum and maximum cardinality of a minimal local isolate dominating set is called the local isolate domination number, denoted by *γlo*(*G*) and the upper isolate domination number, denoted by Γ*lo*(*G*). In this paper, a study on these parameters is initiated.

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# Introduction

By a graph, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [3]. All graphs in this paper are assumed to be non-trivial.

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In a graph *G* = (*V, E*), the degree of a vertex v is defined to be the number of edges incident with v and is denoted by deg v.

The minimum of {*degv*; *v* ∈ *V* (*G*)} is denoted by *δ*(*G*) and the maximum of {*degv*; *v* ∈ *V* (*G*)} is denoted by ∆(*G*).

The open neighbourhood of a vertex *v* ∈ *V* (*G*) is *N* (*v*) = {*u* ∈ *V* (*G*); *uv* ∈ *E*(*G*)} and the closed neighbourhood is *N* [*v*] = *N* (*v*) ∪ {*v*}. The subgraph induced by a set S of vertices of a graph G is denoted by *< S >* with *V* (*< S >*) = *S* and *E*(*< S >*) = {*uv* ∈ *E*(*G*); *u, v* ∈ *S*}. For a set S of vertices, a vertex v is said to be a private neighbour of a vertex *u* ∈ *S* with respect to S if *N* [*v*] ∩ *S* = {*u*}. Furthermore, the private neighbour set of u, with respect to S, *pn*[*u, S*] = {*v*; *N* [*v*] ∩ *S* = {*u*}}. Notice that *u* ∈ *pn*[*u, S*] if u is an isolate in *< S >*, in which case we say that u is its own private neighbour. If G and H are disjoint graphs, then the join of G and H denoted by *G* + *H* is the graph such that *V* (*G* + *H*) = *V* (*G*) ∪ *V* (*H*) and *E*(*G* + *H*) = *E*(*G*) ∪ *E*(*H*) ∪ {*uv*; *u* ∈ *V* (*G*)*, v* ∈ *V* (*H*)}. A wheel on n vertices (*n* ≥ 4), denoted by *Wn*, is the graph *K*1 + *Cn−*1. The vertex corresponding to *K*1 is called the center vertex of *Wn*. The corona of two disjoint graphs *G*1 and *G*2 is defined to be the graph *G* = *G*1 ◦ *G*2 formed from one copy of *G*1 and |*V* (*G*1)| copies of *G*2 where the *ith* vertex of *G*1 is adjacent to every vertex in the *ith* copy of *G*2. Isolate domination in the join and corona of graphs was studied in [1].

The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a detailed survey of domination one can see [4, 5] and [6]. A set D of vertices of a graph G is said to be a dominating set if every vertex in *V* −*D* is adjacent to a vertex in D. A dominating set D is said to be a minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality of a dominating set of a graph G is called the domination number of G and is denoted by *γ*(*G*). The upper domination number Γ(*G*) is the maximum cardinality of a minimal dominating set of G.

A dominating set S of a graph G is said to be an isolate dominating set of G if *< S >* has at least one isolated vertex. An isolate dominating set S is said to be a minimal isolate dominating set if no proper subset of S is an isolate dominating set. The minimum and maximum cardinality of a minimal isolate dominating set of G are called the isolate domination number *γ*0(*G*) and the upper isolate domination number Γ*o*(*G*) respectively. The isolated domination set was introduced by I.Sahul hamid, S.Balamurugan in [7]. An isolate dominating set S is said to be doubly isolate dominating set of G if the subgraph *< V* (*G*) \ *S >* induced by *V* (*G*)*S* has an isolated vertex. The minimum cardinality of the doubly isolate dominating set is called doubly isolate domination number, denoted by *γ00*(*G*). The concept of doubly isolate domination number was introduced by Benjier.H Arriola in 2015 in [2]. This paper introduces such a domination parameter namely local isolate domination number and upper local isolate domination number which are defined as follows.

A dominating set S is said to be local isolate dominating set of G if for each *u* ∈ *S* such that *< N* (*u*) *>* has an isolate vertex. The minimum and maximum cardinality of a minimal local isolate dominating set of G are called the local isolate domination number *γl*0(*G*) and the upper local isolate domination number Γ*lo*(*G*) respectively. When the concept of isolate domination is localized to the neighbour set, we observe that *γ, γo* and *γlo* satisfy all types of relations. This motivated in the study of local isolate domination number of graphs.

# Preliminary Results

**Theorem 2.1** *Let G be a graph of order n ≥ 3 if ∆(G) = n−1 and δ(G) > 1*

then G has no local isolate dominating set.

**Proof :** Let G be a graph of order n ≥ 3. Let v ∈ V (G) such that degG(v) = n − 1, Clearly {v} is a dominating set of G. Since δ(G) > 1, there is no vertex in G which is a pendent. Therefore < N (v) > has no isolate vertex. Thus {v} is not a local isolate dominating set. Let S be a any dominating set of G. If S \ {v} ≠ φ then for each u ∈ S \ {v}, vertex in N(u) is adjacent to v and v is also in N(u). Therefore < N (u) > has no isolate vertex. Thus S \ {v} is not a local isolate dominating set of G. Hence S is not a local isolate dominating set of G.

There are graphs for which local isolate domination set does not exist.

**Corollary 2.2** The local isolate dominating set does not exist for the following graphs:

* 1. *Complete graph Kn.*
  2. *Wheel graph Wn.*
  3. *Fan graph Fn.*

**Observation 2.3** *The local isolate dominating set does not exist for the complete Kn1,n2,…nr , r ≥ 3 graph.*

# Main Results

In this section, the local isolate domination number for some classes of graphs are determined.

**Proposition 3.1**

(i) *For the paths Pn and the cycles Cn we have and *

*(ii) If G is a graph of order n, then 𝛾𝑙𝑜 (𝐺+) = Γ𝑙o(𝐺+) = 𝑛, where 𝐺+ is the graph obtained from G by attaching exactly one edge at every vertex of G.*

***Proof.*** *(i) Obviously and when n≠4, any γ-set of Pn is an local isolate dominating set as well, so that . Every local isolate dominating set is a dominating set so  Thusand so as . Now if Pn = {v1,v2,v3,…vn}then the set S = is a minimal isolate dominating set so that . Further, as any set with more than vertices of Pn can no longer be a minimal isolate dominating, we have  In a similar way one can prove that and *

*(ii) Let S be any minimal isolate dominating set of 𝐺+. Then S must contain each pendant vertex or its neighbour so that S contains at least n vertices. Further, if |𝑆| > 𝑛, then S must contain a pendant vertex together with its support and so 𝑆 − {𝑣}, where v is the support, is an isolate dominating set of 𝐺+, a contradiction to the minimality of S. Hence |𝑆| = 𝑛.*

**Theorem 3.2** *Let G be a graph of order n ≥ 2. Then γlo(G) = 1 if and only if there exists u, v ∈ V (G) such that degG(u) =1 and degG(v) = n − 1.*

**Proof :** *Let G be a graph of order n ≥ 2. Suppose γlo(G) = 1. Let S = {v} be a local isolate dominating set of G. since S is a dominating set and |V (G) \ S| = n − 1, degG(v) = n − 1. Also since S is a γlo-set of G, < N (v) > has an isolate vertex, say u. Therefore u is a pendent vertex of G. Hence degG(u) = 1. Conversely, since there exist a vertex v with degG(v) = n − 1, {v} is a dominating set of G. Since degG(u) = 1, u is a isolate vertex in < N (v) >. Thus γlo(G) = 1.*

**Corollary 3.3** *Let Sn be a star graph of order n ≥ 2, then γlo(Sn) = 1.*

**Proof:** *Let Sn be a star graph of order n ≥ 2. Since Sn contain vertices of degree n − 1 and 1, by the theorem 3.2, γlo(Sn) = 1.*

**Theorem 3.4** If G is a Tree of order n ≥ 2 then G has a local isolate dominating set.

**Proof :** Let G be Tree of order n ≥ 2 and S be any dominating set of G. Suppose G has no local isolate dominating set, there exist a vertex v ∈ S, < N (v) > has no isolate vertex. Thus < N (v) > is connected graph. This implies < N [v] > has a cycle, which is contradiction to G is a tree. Therefore G has a local isolate dominating set.

**Corollary 3.5** *Let T be any tree then γ*(*T* ) = *γo*(*T* ) = *γlo*(*T* )*.*

**Proof:** *Obvious.*

**Theorem 3.6** *Let S be any local isolate dominating set of a graph G and u* ∈ *S. Then there exist a vertex v* ∈ *V* (*G*) *such that uv* ∈ *E*(*G*) *and N* (*u*) ∩ *N* (*v*) = φ*.*

**Proof :** Let S be any local isolate dominating set of a graph G and u ∈ S. Suppose for each v ∈ V (G) such that uv ∈ E(G) and N (u) ∩ N (v) /= φ then there exist a vertex w ∈ N (u) ∩ N (v). This implies w ∈ N (u) and w ∈ N (v). Therefore, < N (u) > has no isolated vertex. Which is contradiction. Therefore there exist a vertex v ∈ V (G) such that uv ∈ E(G) and N (u) ∩ N (v) = φ.

**Theorem 3.7** *For the complete bipartite graph Km,n, γlo*(*Km,n*) = 2*, m* ≥ 2*, n* ≥ 2*.*

**Proof:** Let G be a complete bipartite graph Km,n with at least 4 vertices. Let V (G) = {u1, u2, ..., um} ∪ {v1, v2, ...., vn} then the set S = {ui, vj}, i = 1, 2, ..., m and j = 1, 2, ..., n is a dominating set of G as < N (ui) > and < N (vj) > have isolate vertices. Therefore S is a local isolate dominating set of G. Hence γlo(Km,n) ≤ 2. Since γ(Km,n) = 2 , γlo(Km,n) ≥ 2 . Hence γlo(Km,n) = 2.

If a set has property P, then we say that S is a P-set. A P-set S is a 1-minimal P-set if for any vertex u S, the set S u is not a P-set while it is a minimal P-set if no proper subset of S is a P-set.

Clearly, minimal P-sets are 1-minimal P-sets but not the converse.

**Theorem 3.8** Let S be any local isolate dominating set of a graph G. Then S is minimal if and only if S is 1-minimal.

**Proof :** Let S be a 1-minimal local isolate dominating set of a graph G. Suppose there exists a proper subset S of S that is also an local isolated dominating set of G. Then for all v ∈ S’ , < N (v) > has an isolate vertex. Since S’ is a dominating set, for all vertex in u ∈ S \ S’ is adjacent to at least one vertex in S’ and either u is an isolate vertex in < N (v) >, v ∈ S’ or < N (v) > has an isolated vertex in V \ S.

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**Case (i):** *u is an isolate vertex in < N* (*v*) *>, v* ∈ *S’ then S* \ {*v*} *is local* isolate dominating set of G which is contradiction to 1-minimality of S.

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**Case (ii) :** < N (v) > has a isolated vertex in V \ S. Let w ∈ < N (v) > be isolate vertex in V \ S then S \ {u} is local isolate dominating set of G which is contradiction to 1-minimality of S. Hence S is a minimal. Converse is obvious.

**Theorem 3.9** An local isolate dominating set S of a graph G is minimal if and only if every vertex in S has a private neighbour with respect to S.

**Proof:** Let S be a minimal local isolate dominating set and u be a vertex of S. If u is an isolate in < S > then u is a private neighbour of itself. Suppose u is not an isolate of < S >. If u has no private neighbour with respect to S then the set S \{u} will be a local isolate dominating set. This contradicts the minimality of S and therefore u must have a private neighbour with respect to S. Conversely, suppose S is an local isolate dominating set of G and every vertex of S has a private neighbour with respect to S. We now show that S is a minimal local isolate dominating set. If not, then by Theorem 3.8, S cannot be a 1-minimal dominating set of G and so there is a vertex u in S such that S \ {u} is an local isolate dominating set of G. Therefore, every vertex in V \ (S \ {u}) must have at least one neighbour in S \ {u} and consequently the vertex u can have no private neighbours with respect S. This contradicts our assumption and hence the result follows.

**Corollary 3.10** If S ⊆ V (G) is a local isolate dominating set of G which is minimal with respect to local isolate domination, then S is a minimal dominating set of G.

**Proof.** Let S be a minimal local isolate dominating set. Then by theorem 3.9, every vertex of S has a private neighbour with respect to S and consequently S is a minimal dominating set.

# Join of Graphs

Let A and B be sets which are not necessarily disjoint. The disjoint union of A and B, denoted by *A*∪ *B*, is the set obtained by taking the union of A and B treating each element in A as distinct from each element in B. The union *G*1 ∪ *G*2 of graphs *G*1 and *G*2 with disjoint vertex-sets *V* (*G*1) and *V* (*G*2), respectively, is the graph G where *V* (*G*) = *V* (*G*1)∪ *V* (*G*2) and *E*(*G*) = *E*(*G*1)∪ *E*(*G*2).

**Observation 4.1** *Let G and H be any graphs of order m, n* ≥ 3 *respectively*

with isolate vertex and S be a local isolate dominating set of G + H then

*S* ∩ *V* (*G*) /= φ *and S* ∩ *V* (*H*) /= φ*.*

**Theorem 4.2** Let G and H be any graphs. Then S ⊆ V (G + H) is a local isolate dominating set of G + H if and only if G and H have isolated vertices.

**Proof.** Let G and H be any graphs and S ⊆ V (G + H) be a local isolate dominating set of G + H. Suppose G and H have no isolated vertex. For each u ∈ S, < N (u) > is connected, which is contradiction. Therefore there are isolated vertex in both G and H.

Conversely, u and v be isolated vertices of G and H respectively, then S = {u, v} is a dominating set of G + H and also N (u) ≥ V (H) and N (v) ≥ V (G). Thus < N (u) > and < N (v) > have isolated vertex.

Therefore S is local isolate dominating set of G + H.

**Corollary 4.3** *Let G and H be any graphs with isolated vertex, then*

*γlo*(*G* + *H*) ≤ 2*.*

**Proof.** *Let G and H be graphs with isolated vertex. Suppose either G* = *K*1 *or*

*H* = *K*1 *or G* = *H* = *K*1 *then clearly, γlo*(*G* + *H*) = 1*. Suppose G* /= *K*1 *and H* /= *K*1 *by the theorem 4.2, γlo*(*G* + *H*) = 2*. Thus γlo*(*G* + *H*) ≤ 2*.*

# Conclusion

In this paper the concept of local isolate domination number for graphs is introduced and a study on this notion is initiated. Also local isolate domination parameters for some classes of graphs are determined.

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