Exploring the property of Rascal Triangle

Suryaprakash Vishwakarma
suryaprakash.vishwakarma703@gmail.com
B.K. Birla College(Autonomous) Kalyan, Mumbai
Dr,Seema Purohit
supurohit@gmail.com
B.K. Birla College(Autonomous) Kalyan, Mumbai

July 2023

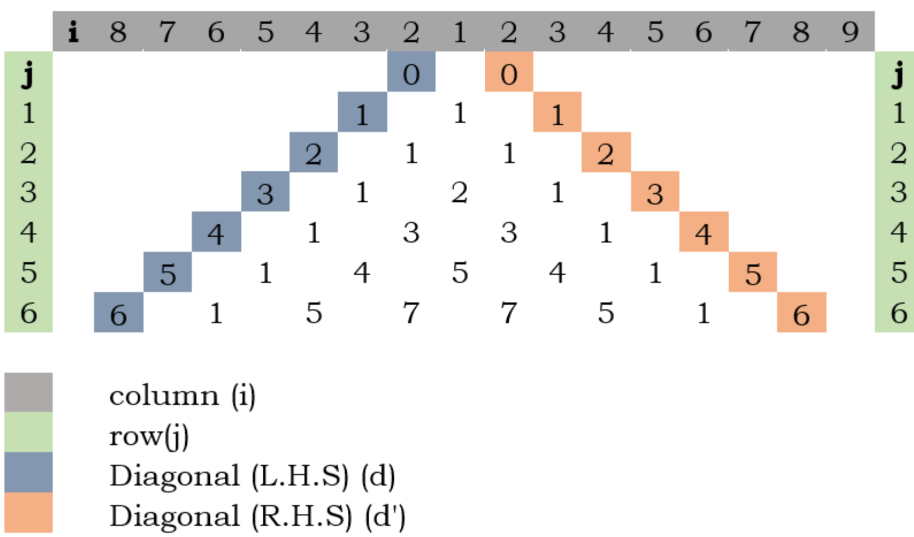

Figure 1: RASCAL TRIANGLE

1 Construction of the Rascal triangle.

The *Rascal triangle* is classified in to row, column and diagonals as shown in fig 1 which are denoted by j,i and $d(LHS)$ and $d'(RHS)$ respectively. we can construct the triangle by two different method which are *Row wise construction and Column wise construction.* We are going to see hear both the method one by one.

1. Row wise construction.

2. column wise construction.

1.1 Row wise construction.

We can construct the triangle row wise denoted by $E_R(j,n)(n^{th}$ element of row j). More generally, all entries of j^{th} row can be constructed using the following recursive definition ,

$$E_R(j,1) = 1 \ldots \forall j \in \mathbb{N}, \quad and$$

$$E_R(j,k+1) = E_R(j,k) + j - 2k \ldots \forall j,k \in \mathbb{N}$$

Now we have to find the general formula to construct the whole row for that we are going to take all the entries of any row from our *Rascal Triangle (fig.1)*.
let's take 7^{th} row($j = 7$), and the entries are,

1	6	9	10	9	6	1

$n(n^{th}entry)$	element		In general	
1	1	$= 1$	$= 1$	$= 1$
2	6	$= 1 + (7 - 2)$	$= 1 + (j - 2)$	$= j - 2 + 1$
3	9	$= 6 + (7 - 4)$	$= j - 2 + 1 + (j - 4)$	$= 2j - 6 + 1$
4	10	$= 9 + (7 - 6)$	$= 2j - 6 + 1 + (j - 6)$	$= 3j - 12 + 1$
5	9	$= 10 + (7 - 8)$	$= 3j - 12 + 1 + (j - 8)$	$= 4j - 20 + 1$
6	6	$= 9 + (7 - 10)$	$= 4j - 20 + 1 + (j - 10)$	$= 5j - 30 + 1$
7	1	$= 6 + (7 - 12)$	$= 5j - 30 + 1 + (j - 12)$	$= 6j - 42 + 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Similarly we can go up to n for any j, for more generalisation we are going to take the general form of an element/entry from the above table.

let's take general form of $5^{th}(n = 5)$ entry/element,

$$E_R(7,5) = 4j - 20 + 1$$

$$E_R(7,5) = 4j - 4 \times 5 + 1$$
$$E_R(7,5) = (5-1)j - (5-1)5 + 1$$

Thus, more generally we can write this in terms of n and j,

$$\therefore E_R(j,n) = (n-1)j - (n-1)n + 1$$

$$\therefore E_R(j,n) = (n-1)(j-n) + 1 \ldots (1 \leq n \leq j)$$

Hear we got the general formula to get n^{th} entry/element of row j but we have to prove this, and we can prove this formula by method of induction.

Proof(by method of induction)...

we have,

$$E_R(j,n) = (n-1)(j-n) + 1 \ldots (1 \leq n \leq j) \ldots\ldots (a)$$

We prove $eq(a)$ for $n = 1$.
We know that,

$$L.H.S. = E_R(j,1) = 1 \ldots\ldots (from \ \ definition)$$

$$R.H.S. = 0 \times (j-1) + 1 = 1$$

Hence, $eq(a)$ is true for $n = 1$.
Let us assume that (a) is true for $n = k$.That is,

$$E_R(j,k) = (k-1)(j-k) + 1 \ldots (1 \leq k \leq j) \ldots\ldots (b)$$

To prove that,

$$E_R(j,k+1) = (k+1-1)(j-k-1) + 1$$

$$E_R(j, k+1) = k(j-k-1)+1 \ldots\ldots (c)$$

Now consider L.H.S.,

$$E_R(j, k+1) = E_R(j, k)+j-2k \ldots\ldots [from\ definition\ of\ the\ rascal\ triangle(rowwise)]$$

$$E_R(j, k+1) = (k-1)(j-k)+1+j-2k \ldots\ldots [from\ \ eq(b)]$$

$$E_R(j, k+1) = kj - k^2 - j + k + 1 + j - 2k$$

$$E_R(j, k+1) = kj - k^2 - k + 1$$

$$E_R(j, k+1) = k(j-k-1)+1$$

$$= eq(c)$$

Hence proved...

1.2 Column wise construction.

We can construct the triangle column wise denoted by $E_C(i,n)$(n^{th} element of Column i). More generally, all entries of i^{th} column can be constructed using the following recursive definition ,

$$E_C(i,1) = 1 \ldots \forall i \in \mathbb{N}, \quad and$$

$$E_C(i,k+1) = E_C(i,k) + i + 2(k-1) \ldots \forall j, i \in \mathbb{N}$$

Now we have to find the general formula to construct the whole column, for that we are going to take some of the entries(because column doesn't have end element like low) of any column from our *Rascal Triangle (fig.1)*.
let's take 2^{th} column($i=2$), and the entries are,

1	3	7	13	21	21	31	43	57......

Now observe,

$n(n^{th}entry)$	element		In general	
1	1	$=1$	$=1$	$=1$
2	3	$=1+(2+0)$	$=1+(i+0)$	$=i+1$
3	7	$=3+(2+2)$	$=i+1+(i+2)$	$=2i+2+1$
4	13	$=7+(2+4)$	$=2i+2+1+(i+4)$	$=3i+6+1$
5	21	$=13+(2+6)$	$=3i+6+1+(i+6)$	$=4i+12+1$
6	31	$=21+(2+8)$	$=4i+12+1+(i+8)$	$=5i+20+1$
7	43	$=31+(2+10)$	$=5i+20+1+(i+10)$	$=6i+30+1$
8	57	$=43+(2+12)$	$=6i+30+1+(i+12)$	$=7i+42+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Similarly we can go up to n for any i, for more generalisation we are going to take the general form of an element/entry from the above table.

let's take general form of $7^{th}(n=7)$ entry/element,

$$E_C(2,7) = 6i + 30 + 1$$

$$E_C(2,7) = 6i + 6 \times 5 + 1$$

$$E_C(2,7) = (7-1)i + (7-1)(7-2) + 1$$

Thus, more generally we can write this in terms of n and i,

$$\therefore E_C(i,n) = (n-1)i + (n-1)(n-2) + 1$$

$$\therefore E_C(i,n) = (n-1)(i+n-2) + 1 \ldots (i, n \in \mathbb{N})$$

Hear we got the general formula to get n^{th} entry/element of column i but we have to prove this, and we can prove this formula by method of induction.

Proof(by method of induction)...

we have,

$$E_C(i,n) = (n-1)(i+n-2) + 1 \ldots (\forall i, n \in \mathbb{N}) \ldots\ldots (a)$$

We prove $eq(a)$ for $n = 1$.
We know that,

$$L.H.S. = E_C(i,1) = 1 \ldots\ldots (from \;\; definition)$$

$$R.H.S. = 0 \times (i+1-1) + 1 = 1 \ldots (\forall i \in \mathbb{N})$$

Hence, $eq(a)$ is true for $n = 1$.
Let us assume that (a) is true for $n = k$.That is,

$$E_C(i,k) = (k-1)(i+k-2) + 1 \ldots (\forall i, k \in \mathbb{N}) \ldots\ldots (b)$$

To prove that,

$$E_C(i,k+1) = (k+1-1)(i+k+1-2) + 1$$

$$E_C(i,k+1) = k(i+k-1) + 1 \ldots\ldots (c)$$

Now consider L.H.S.,

$$E_C(i,k+1) = E_C(i,k) + i + 2(k-1) \ldots [from \; definition \; of \; the \; rascal \; triangle(column \; wise)]$$

$$E_C(i,k+1) = (k-1)(i+k-2) + 1 + i + 2(k-1) \ldots\ldots [from \;\; eq(b)]$$

$$E_C(i,k+1) = (k-1)(i+k-2+2) + 1 + i$$

$$E_C(i,k+1) = (k-1)(i+k) + 1 + i$$

$$E_C(i, k+1) = ki + k^2 - i - k + 1 + i$$
$$E_C(i, k+1) = ki + k^2 - k + 1$$
$$E_C(i, k+1) = k(i + k - 1) + 1$$
$$= eq(c)$$

Hence proved...

2 INTERSECTION...

As we know in the *Rascal Triangle* we have some different type of notation or representation like *Row(j), Column(i),Diagonal from LHS(d) and Diagonal from RHS(d')*. Hear we are going to introduce you a new concept INTERSECTION. We can clearly observe that when we follow any 2 of them then they intersect each other at a point(on an element) under certain condition. Now hear we are going to see the intersectional element(where they intersect each other) and there condition also.

1. Row and column.
2. Row and Diagonal.
3. Column and Digonal.

2.1 Row and Column...

In the *Rascal Triangle* if we observe carefully we can see that when we go along Row and Column simultaneously then they intersect each other on an element denoted by $E_{CR}(i, j)$ and hear we are going to find that element.

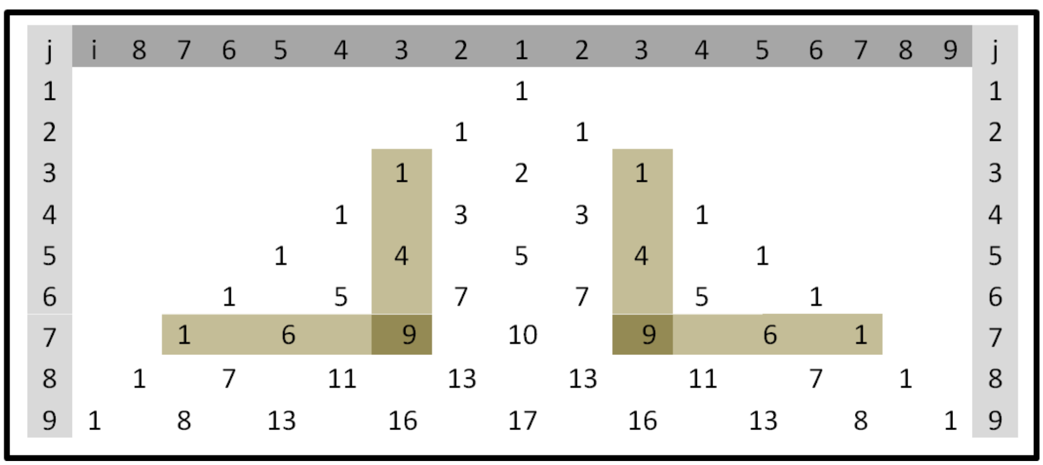

Figure 2: Intersectional element(Row and column)

Note: When i and j have same parity and $i \leq j$ only there the element exist and on different parity element doesn't exist.
Where i and j intersect each other for that element n is same for both $E_C(i,n)$ and $E_R(j,n)$ and $E_C(i,n) = E_R(j,n)$

$$\because E_C(i,n) = E_R(j,n)$$
$$\therefore (n-1)(i+n-2)+1 = (n-1)(j-n)+1$$
$$\therefore (i+n-2) = (j-n)$$
$$\therefore 2n = j-i+2$$
$$\therefore n = \frac{j-i}{2} + 1 \ldots\ldots (\star)$$

Now,

$$E_{CR}(i,j) = \frac{E_C(i,n)+E_R(j,n)}{2}$$
$$E_{CR}(i,j) = \frac{(n-1)(i+n-2)+1+(n-1)(j-n)+1}{2}$$
$$E_{CR}(i,j) = \frac{(n-1)(i+n-2+j-n)+2}{2}$$
$$E_{CR}(i,j) = \frac{(n-1)(i+j-2)+2}{2} \ldots (eq(a))$$
$$E_{CR}(i,j) = \frac{\left(\frac{j-i}{2}+1-1\right)(i+j-2)+2}{2} \ldots (from \quad \star \quad and \quad eq(a))$$
$$E_{CR}(i,j) = \frac{\left(\frac{j-i}{2}\right)(i+j-2)+2}{2}$$
$$E_{CR}(i,j) = \frac{(j-i)(j+i-2)+4}{4} \ldots [j \geq i] \ldots [\star\star]$$

We can find this $E_{CR}(i,j)$(intersectional element) with two more ways which are given as,

1. With the help of Row$[E_C(i,n)]$
2. With the help of column$[E_R(j,n)]$

2.1.1 With the help of Row$[E_C(i,n)]$.

We know that,

$$E_R(j,n) = (n-1)(j-n) + 1 \ldots (c)$$

$$E_R\left(j, \frac{j-i}{2}+1\right) = \left(\frac{j-i}{2}+1-1\right)\left(j-\frac{j-i}{2}-1\right)+1 \ldots [from \star \ and \ eq(c)]$$

$$E_C\left(j, \frac{j-i}{2}+1\right) = \left(\frac{j-i}{2}\right)\left(\frac{2j-j+i-2}{2}\right)+1$$

$$E_C\left(j, \frac{j-i}{2}+1\right) = \left(\frac{j-i}{2}\right)\left(\frac{j+i-2}{2}\right)+1$$

$$E_C\left(j, \frac{j-i}{2}+1\right) = \left(\frac{(j-i)(j+i-2)+4}{4}\right)$$

$$= \star\star$$

$$\therefore E_R\left(j, \frac{j-1}{2}+1\right) = E_{CR}(i,j) = \frac{(j-i)(j+i-2)+4}{4} \ldots [j \geq i]$$

Hence proved...

2.1.2 With the help of Column$[E_R(j,n)]$.

We know that,

$$E_C(i,n) = (n-1)(i+n-2) + 1 \ldots (b)$$

$$E_C\left(i, \frac{j-i}{2}+1\right) = \left(\frac{j-i}{2}+1-1\right)\left(i+\frac{j-i}{2}+1-2\right)+1 \ldots [from \star \ and \ eq(b)]$$

$$E_C\left(i, \frac{j-i}{2}+1\right) = \left(\frac{j-i}{2}\right)\left(\frac{2i+j-i-2}{2}\right)+1$$

$$E_C\left(i, \frac{j-i}{2}+1\right) = \left(\frac{j-i}{2}\right)\left(\frac{2i+j-i-2}{2}\right)+1$$

$$E_C\left(i, \frac{j-i}{2}+1\right) = \frac{(j-i)(j+i-2)+4}{4}$$

$$= \star\star$$

$$\therefore E_C\left(i, \frac{j-1}{2}+1\right) = E_{CR}(i,j) = \frac{(j-i)(j+i-2)+4}{4} \ldots [j \geq i]$$

Hence proved...

2.2 Row and Diagonal...

intersectional element of row and diagonal.denoted by $E_{RD}(j,d)$ and we can prove this by two method which are given below.

1. With the help of $E_R(j,n)$
2. With the help of $E_D(d,n)$

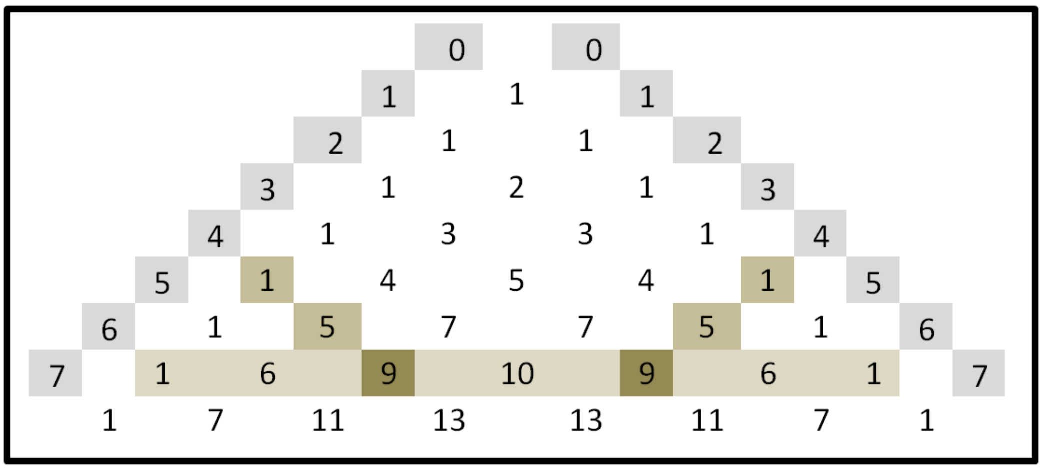

Figure 3: Intersectional element(Row and Digonal)

$\because$ n is same for $E_R(j,n)$ and $E_D(d,n)$

$$\therefore (n-1)(j-n)+1 = 1+(n-1)d$$

$$\therefore (n-1)(j-n) = (n-1)d$$

$$\therefore j-n = d$$

$$\therefore n = j-d \ldots [a]$$

2.2.1 With the help of $E_R(j,n)$

We know that,

$$E_R(j,n) = (n-1)(j-n)+1$$

$$\therefore E_R(j,j-d) = (j-d-1)(j-j+d)+1 \ldots [from\ eq(a)]$$

$$E_{RD}(j,d) = (j-d-1)d+1$$

$$\therefore E_{RD}(j,d) = (j-d-1)d+1 \ldots [just\ a\ different\ notation]$$

2.2.2 With the help of $E_D(d, n)$ *or* t_n

We know that,

$$t_n = E_D(d, n) = a + (n - 1)d$$
$$\therefore E_D(d, j - d) = a + (j - d - 1)d \ldots [from\ eq(a)]$$
$$\therefore E_{RD}(j, d) = 1 + (j - d - 1)d$$
$$\therefore E_{RD}(j, d) = (j - d - 1)d + 1$$

2.3 Column and Diagonal

Intersectional element if row and diagonal, denoted by E_{CD}. hear we will get two different intersectional element according to the side of the diagonal and column.

1. Both are on same side $[E_{CD}(i, d)]$
2. Both are on opposite side $[E_{CD'}(i, d)]$

2.3.1 Both are on same side $[E_{CD}(i, d)]$

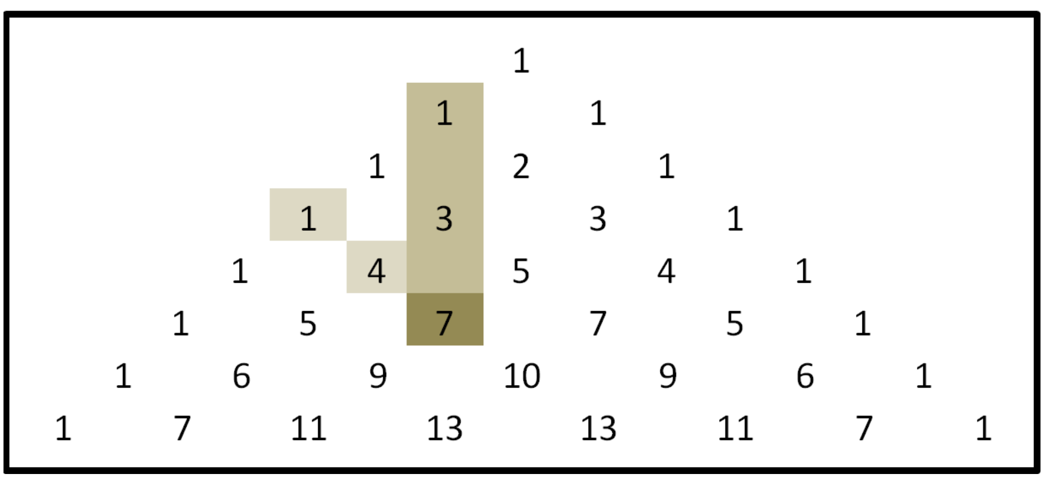

Figure 4: Intersectional element [Column and Diagonal(same side)]

if $i \leq d + 1$ then $E_{CD}(i, d) \ldots (i \in \mathbb{N}, d \in \mathbb{W})$ will exist...
the element where column i and diagonal d intersect each other $E_{CD}(i, d)$,

when they are on same side for that element n is same for both $E_C(i,n)$ and $E_D(d,n)$.

$$\therefore E_C(i,n) = E_D(d,n)$$
$$(n-1)(i+n-2)+1 = 1+(n-1)d$$
$$(n-1)(i+n-2) = (n-1)d$$
$$i+n-2 = d$$
$$n = d-i+2 \ldots (\star)$$

Now,

$$E_{CD}(i,d) = \frac{E_C(i,n)+E_D(d,n)}{2}$$
$$E_{CD}(i,d) = \frac{(n-1)(i+n-2)+1+1+(n-1)d}{2}$$
$$E_{CD}(i,d) = \frac{(n-1)(i+n-2)+(n-1)d+2}{2}$$
$$E_{CD}(i,d) = \frac{(n-1)(i+n+d-2)+2}{2}$$
$$E_{CD}(i,d) = \frac{(d-i+2-1)(i+d-i+2+d-2)+2}{2} \ldots [from \star]$$
$$E_{CD}(i,d) = \frac{(d-i+1)2d}{2}+1$$
$$E_{CD}(i,d) = (d-i+1)d+1 \ldots [\star\star]$$

We can find this $E_{CD}(i,d)$(intersectional element of Row and Diagonal) with two more ways which are given as,

1. With the help of Column[$E_C(i,n)$]
2. With the help of Diagonal[$E_D(d,n)$]

1. With the help of Column[$E_C(i,n)$]

. We know that,

$$E_C(i,n) = (n-1)(i+n-2)+1 \ldots (b)$$
$$E_C(i,d-i+2) = (d-i+2-1)(i+d-i+2-2)+1 \ldots (b)$$
$$E_{CD}(i,d) = (d-i+1)d+1 \ldots [from\ eq(c)\ and\ (\star)]$$
$$= \star\star$$

Hence proved...

2. With the help of Diagonal$[E_D(d,n)]$

We know that,

$$E_D(d,n) = 1 + (n-1)d \ldots (c)$$

$$E_D(d, d-i+2) = 1 + (d-i+2-1)d \ldots [from\ eq(c)\ and\ (\star)]$$

$$E_{CD}(i,d) = (d-i+1)d + 1$$

$$= \star\star$$

Hence proved...

2.3.2 Both are in opposite side[$\mathbf{E}_{CD}(i,d)$]

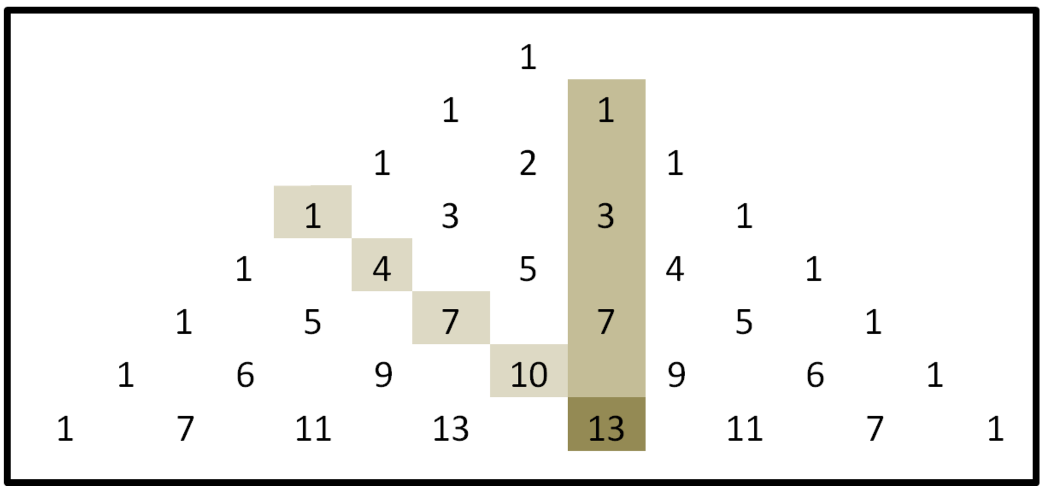

Figure 5: Intersectional element(Column and Diagonal[Opposite])

$E_{CD}(i,d)$ axist always $i \in \mathbb{N}, i \geq 2, d \in \mathbb{W}$ the element where diagonal and column intersect eachother and they are on opoosite side ,denoted by $E_{CD'}(i,d)$

We can find the $E_{CD}(i,d)$ with teow different method which are given bellow.

With the help of Column$[E_C(i,n)]$ With the help of Diagonal$[E_D(d,n)]$

1. With the help of Column$[E_C(i, n)]$

the intersectional element of diagonal and column which are in opposite side have different value of n for both column and diagonal. hear the relation between n, i, d,

$$n = d + 1$$

We know that,

$$E_C(i, n) = (n - 1)(i + n - 2) + 1$$

$$E_C(i, d + 1) = (d + 1 - 1)(i + d + 1 - 2) + 1$$

$$E_{CD'}(i, d) = (i + d - 1)d + 1$$

2. With the help of Diagonal$[E_D(d, n)]$

the intersectional element of diagonal and column which are in opposite side have different value of n for both column and diagonal. hear the relation between n, i, d,

$$n = i + d$$

We know that,

$$E_D(d, n) = 1 + (n - 1)d$$

$$E_D(d, i + d) = 1 + (i + d - 1)d$$

$$E_{CD}(i, d) = (i + d - 1)d + 1$$

3 SUMATION...

1. Sum of first $'n'$ element in Column $'i'$.
2. Sum of first $'n'$ element in Row $'j'$.
3. Sum of all element of Row $'j'$.
4. Sum of all element from Row 1 to Row $'j'$.

3.1 Sum of first $'n'$ element in Column $'i'$.

we know the general formula for column $E_C(i,n)$.

$$E_C(i,n) = (n-1)(i+n-2)+1)$$

$$E_C(i,n) = (n-1)i+(n-1)(n-2)+1)$$

$$S_C(i,n) = \sum_{n=1}^{n} E_C(i,n)$$

$$S_C(i,n) = (0i+0+1)+(1i+0+1)+(2i+2+1)+(3i+6+1)+$$

$$(4i+12+1)+(5i+20+1)+\ldots+((n-1)i+(n-1)(n-2)+1$$

$$S_C(i,n) = [0i+1i+2i+3i+4i+5i+\ldots+(n-1)i]+[0+0+2+6+12+$$

$$20+\ldots(n-1)(n-2)]+[1+1+1+1+1+1+\ldots+1]$$

$$S_C(i,n) = [i(1+2+3+4+5+\ldots+(n-1)]+[(1^1+1)+(2^2+2)+(3^3+3)+$$

$$\ldots+(n-2)^2+(n-2)]+[1+1+1+1+1+1+\ldots+1]$$

$$S_C(i,n) = \left(i\sum_{n=1}^{n-1} n\right)+\left(\sum_{n=1}^{n-2}(n^2+n)\right)+n$$

$$S_C(i,n) = \frac{n(n-1)}{2}i+\frac{n(n-1)(n-2)}{3}+n$$

3.2 Sum of first $'n'$ element in Row $'j'$.

We know the general formula for Row$E_R(n, j)$.

$$E_R(j, n) = (n-1)(j-n) + 1$$

$$E_R(j, n) = (n-1)j - n(n-1) + 1$$

$$S_R(j, n) = \sum_{n=1}^{n} E_R(j, n)$$

$$S_R(j, n) = (0j-0+1)+(1j-2+1)+(2j-6+1)+(3j-12+1)+(4j-20+1)++(5j-30+1)$$

$$S_R(j, n) = (0j+1j+2j+3j+\ldots+(n-1)j)-(0+2+6+12+\ldots+n(n-1))+(1+1+1+1+\ldots+1)$$

$$S_R(j, n) = j(0+1+2+3+\ldots+(n-1))-((0^2+0)+(1^2+1)+(2^2+2)+(3^2+3)+ \ldots + n(n-1)) + (1+1+1+1+\ldots+1)$$

$$S_R(j, n) = \left(j\sum_{n=1}^{n-1} n\right) - \left(\sum_{n=1}^{n-1}(n^2+n)\right) + n$$

$$S_R(j, n) = j\left(\frac{n(n-1)}{2}\right) - \left(\frac{n(n-1)(n+1)}{3}\right) + n$$

3.3 Sum of all element of row $'j'$.

Note... In any row 'j' , the value of 'j' and the total no. of element present in that row is always same ....

$$\therefore (n = j)$$

We know that,

$$S_R(j, n) = j\left(\frac{n(n-1)}{2}\right) - \left(\frac{n(n-1)(n+1)}{3}\right) + n$$

$$S_R(j, j) = j\left(\frac{j(j-1)}{2}\right) - \left(\frac{j(j-1)(j+1)}{3}\right) + j$$

$$S_R(J) = j(j-1)\left(\frac{j}{2} - \frac{(j+1)}{3}\right)$$

$$S_R(J) = \frac{j(j-1)(3j-2j-2)}{6}$$

$$S_R(J) = \frac{j(j-1)(j-2)}{6}$$

3.4 Sum of all element from Row '1' to Row $'j'$ ($\mathbb{S}(j)$).

We know the sum off all element of any Row $'j'$, which is given as.

$$S_R(J) = \frac{j(j-1)(j-2)}{6}$$

$$\therefore \mathbb{S}(j) = \sum_{j=1}^{j} \frac{j(j-1)(j-2)}{6}$$