**Modeling of Gauss Elimination Technique for Game Theory**

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**ABSTRACT**

In this chapter, a novel approach named Gauss Elimination Technique (GET) is proposed to find the optimal solution of game theoretical problems. It is based upon the concept of bounds. To do so, we formulate two Linear Programming Problems (LPP) for both players by the given game problem. By applying the Gauss Elimination Technique separately to both the linear programming problems, one can get the optimal solution of both the LPP’s and as well game problem. The proposed technique is very simple to understand the calculation as compared to earlier Simplex Method for LPP.

**Keywords:** Optimal solution**,** Linear programming problem, Gauss Elimination Technique.

**1. INTRODUCTION**

In operation research, mathematical modeling is very easy way to describe any phenomenon or problem. Game theory is a theoretical framework for conceiving social scenarios among competing players. It is the science of strategy. The intension of game theory is to layout various situations and predict their most likely outcomes in variety of fields. Game theory explains the strategic action of two or more than two competitors in a given situation. The main focus of game theory is the game, in which one player’s payoff is contingent on the strategy implemented by the other player. There are some important real-world situations which are competitive in nature such as labour-management negotiations, elections and voting, agricultural crop selection, stock market, military conflicts, bidding at auctions etc. Game theoretical model has great potential to analyze above mentioned situations. This mathematical modeling provides perception, guidance and solution of a problem, which is very efficient and accurate in nature.

In numerical analysis, Elimination is very important method to find out numerical solution of different type of problems. Basically, elimination techniques are used in engineering streams and applied sciences. Elimination technique is a new algorithm for solve game theoretical problem.

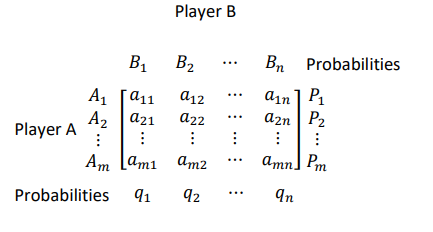
**2. Gauss Elimination Technique for Game Problem**

Gauss elimination technique is basically use in numerical analysis for finding the solution of a system of linear equations with ‘n’ variables and ‘n’ equations. Elimination method eliminates decision variables or unknowns of system of linear equations one by one. Therefore, the matrix of coefficients of the system of equations transforms to an upper triangular matrix. At last, there remains only one equality which has one variable. After that, we calculate value of other variables by back substitution process. By solving equations, we get a single solution or value but when inequalities have solved then we find out more than one possibility for solution in bounded form. From these bounded values we have to select minimum value or maximum value according to the objective function of given problem.

In this chapter, Gauss elimination technique has been applied for a game problem, for this we formulate LPP of game problem. Now for apply Gauss elimination technique on this LPP of game problem, we have to convert this LPP into standard form. To do so, the objective function will be treated as constraint and the sign of all inequalities will be same. “i.e., sign of objective function, constraints and non-negative restrictions” will be same. After combination of inequalities, variables eliminate one by one. i.e., in every iteration one variable and one inequality decrease. So, at final we get some inequalities with single variable, from which we obtain value of that single variable in bounded form according to the object for a player of game problem.

**3. Problem Formulation for Gauss Elimination Technique**

The general form of game problem can be written as:

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This general form delineated a m×n payoff matrix for any two-person game problem. Here, player A has m strategies (*A1,A2,……,Am*) and player B has n strategies (*B1,B2,……,Bn*). *p1,p2,……,pm* are probabilities to choose strategies *A1,A2,……,Am* respectively for player A and *q1,q2,……,qn* are probabilities to choose strategies *B1,B2,……,Bn* respectively for player B. Each entry of payoff matrix shows payoff corresponding to chosen strategy by player A and player B. For example, shows payoff corresponding to strategy *A1* and strategy *B1* chosen by player A and player B respectively.

Now, we formulate the game problem for Gauss elimination technique. Let *V* be the value of game or well said expected gain or loss. Here, we consider that Player B’s objective is to minimize the expected loss, which can be achieved by minimizing V. The expected losses for player B can written in the form of linear combination with the help of payoff matrix given for any game problem.

So, the expected losses for player B will be as follows:

Dividing the above constraints by V, we get

+ + + 1

+ + + 1

+ + + 1

For simplification, take = ; j= 1, 2…...n.

In order to minimize V, player B can maximize .

Since

Divide above equation by V. We get,

+ + + =

Let = So, = = .

**LPP form of game problem for player B:**

Max. = =

Subject to

and ≥ 0; j = 1, 2…...n.

where = ; j= 1, 2…...n.

**LPP form of game problem for player A:**

Min. = =

Subject to

and ; i = 1, 2……m.

where = ; i = 1, 2……m.

Now for solution of game problem by Gauss elimination technique, we have to convert the LPP form of game problem into a standard form. The standard form for Gauss elimination technique contains either equations or inequalities with same sign. So, we convert objective function into inequality. Also, ensure that sign of inequalities is same for converted objective function, constraints and restricted variables.

**Standard form of game problem for Gauss elimination technique:**

**For player B**

Max.

; j = 1, 2……, n.

where, = ; j= 1, 2…...n.

**For player A**

Min.

; i = 1, 2……m.

where, = ; i = 1, 2……m.

Now, we combine all inequalities in such a way that the inequalities and variables must reduce one by one in each iteration. We apply GET on both LPPs’ one by one.

**4. Numerical Illustration**

**1.** Solve the following game problem: Player B

PlayerA

**Solution:** Let us assume that *p1* & *p2*are probabilities for player A and *q1*& *q2* are probabilities for player B to choose strategies for maximum gain of player A and minimum loss of player B.

**Formulation of LPP for player B:**

Let *V* be the maximum gain of player A. Then for minimum loss of player B we have to minimize *V*. So, objective function and constraints for player B are

Min. *V*

Subject to *q1 +* 7*q2 V*

6*q1 +* 2*q2 V*

*q1 + q2 =* 1

Divide above inequalities and equation by V. Then,

Let y1 = and y2 = .

Also, min. *V* = max. = max. *zq*

So,

Max. *zq = y1 + y2*

Subject to  *y1*+ 7*y2* ≤ 1

6*y1*+ 2*y2* ≤ 1

and *y1,y2* ≥ 0

Making standard form by treating objective function as constraint and all inequalities of same sign for Gauss elimination technique, we have

Max. *zq*

*zq - y1 - y2 ≤* 0… (4.1)

*y1 +* 7*y2 ≤* 1… (4.2)

6*y1 +* 2*y2 ≤* 1… (4.3)

*- y1 ≤* 0… (4.4)

*- y2 ≤* 0… (4.5)

In the first stage of Gauss elimination technique with the help of (4.1), we obtain

Max. *zq*

*zq* + 6*y2* ≤ 1 … (4.6)

6*zq* - 4*y2* ≤ 1 … (4.7)

*zq* - *y2* ≤ 0… (4.8)

- *y2* ≤ 0 … (4.9)

In the second stage of Gauss elimination technique with the help of (4.6), we have

Max. *zq*

*zq* ≤  … (4.10)

*zq* ≤ … (4.11)

*zq* ≤ 1 … (4.12)

Here, our object is to maximize *zq*, so select maximum value of *zq* from the bounded values of *zq.* Values of *zq* from inequality (4.10), (4.11) and (4.12) are {}.

Hence, max. *zq* = max. {} = 1.

But *zq*= 1 does not satisfy inequalities (4.6) to (4.9) altogether. i.e., *zq* = 1 gives *y2* ≤ 0 Hence, we select next maximum value of *zq* which is less than 1, we have *zq* = . Now, putting *zq* = in inequalities (4.6) to (4.9), we get different bounded values for variable *y2* out of this *y2*= is the only value that satisfies all inequalities (4.6) to (4.9) altogether. Hence *y2* = .

Now, putting *zq* = and *y2* = in the inequalities (4.1) to (4.5), we get different bounded values for variable *y1* out of this *y1* = is the only value that satisfies all the inequalities (4.1) to (4.5) altogether. Hence *y1* = .

So, *zq* = implies that = and *V* = 4.

Also, *y1* = implies that *q*1 =

and *y2* = implies that *q*2 = .

Hence, strategies for players B (, )

and value of game = 4.

**Formulation of LPP for player A:**

LPP for player A is dual of LPP of player B. So, we have

Min. *zp = x1 + x2*

Subject to *x1* + 6*x2* ≥ 1

7*x1* + 2*x2* ≥ 1

and *x1, x2* ≥ 0

where *x1* = and *x2* = .

Again, we make standard form for apply Gauss elimination technique on above LPP. We have

Min. *zp*

*zp - x1 - x2* ≥ 0 … (4.13)

*x1 +* 6*x2*≥ 1 … (4.14)

7*x1* + 2*x2*≥ 1 … (4.15)

*x1* ≥ 0 … (4.16)

*x2* ≥ 0… (4.17)

In the first stage of gauss elimination with the help of (4.13), we have

Min. *zp*

*zp* + 5*x2* ≥ 1 … (4.18)

7*zp* – 5*x2* ≥ 1 … (4.19)

*zp* - *x2* ≥ 0 … (4.20)

*x2* ≥ 0 … (4.21)

In the second stage of Gauss elimination with the help of (4.18), we have

Min. *zp*

*zp*≥ … (4.22)

*zp*≥ … (4.23)

*zp* ≥1 … (4.24)

Here, our object is to minimize *zp*, so select minimum value of *zp* from the bounded values of *zp*.

Hence, min. *zp*= min {, , 1} = .

Now putting *zp*= in inequalities (4.18) to (4.21). But these inequalities are not satisfied by *zp*= i.e., *zp* = gives different bounded values of *x2* and none of them is satisfy inequalities (4.18) to (4.21) simultaneously. So, we select next minimum value of *zp*that is greater than , so *zp*= .

Again, putting *zp* = in inequalities (4.18) to (4.21), we get different bounded values for variable *x2*out of this *x2*= is the only value that satisfies all the inequalities (4.18) to (4.21) altogether, hence *x2*=

Now putting *zp* = and *x2* = in inequalities (4.13) to (4.17), we get different bounded values for variable *x1*, out of this *x1* = is the only value that satisfies all the inequalities (4.13) to (4.17) altogether. Hence *x1* = .

Hence *zp* = implies that = and *V* = 4.

Also, *x1* = implies that *p*1 = and *x2* = implies that *p*2 =

So, strategies for player A ( , )

and value of game = 4.

Hence the optimal solution of game problem by Gauss elimination technique is: -

value of game = 4.

strategies for player A ( , ).

strategies for player B (, ).

**2.** Solve the following game problem:

Player B

PlayerA

**Solution:** Let us assume that *p1,* *p2*, *p3* & *p4* are probabilities for player A and *q1* & *q2* are probabilities for player B to choose strategies for maximum gain of player A and minimum loss of player B.

**Formulation of LPP for player B:**

Let *V* be the maximum gain of player A. Then for minimum loss of player B we have to minimize *V*. So, objective function and constraints for player B are

Min. *V*

Subject to -2*q1  V*

3*q1 - q2 V*

-3*q1 +* 2*q2 V*

5*q1 -* 4*q2 V*

*q1 + q2 =* 1

Divide above inequalities and equation by V. Then,

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Let *y1 =*  and *y2 = .* Also, min. *V* = max. = max. *zq.* So,

Max. *zq = y1 + y2*

Subject to -2*y1*≤ 1

3*y1*- *y2* ≤ 1

-3*y1*+ 2*y2* ≤ 1

5*y1*- 4*y2* ≤ 1

and *y1,y2* ≥ 0

Make the standard form by treating objective function as constraint and all inequalities of same sign for Gauss elimination technique, we have

Max. *zq*

*zq* - *y1* - *y2 ≤* 0… (4.25)

-2*y1 ≤* 1… (4.26)

3*y1* - *y2 ≤* 1… (4.27)

-3*y1* + 2*y2 ≤* 1… (4.28)

5*y1* - 4*y2 ≤* 1 … (4.29)

- *y1 ≤* 0… (4.30)

- *y2 ≤* 0… (4.31)

In the first stage of Gauss elimination technique with the help of (4.25), we obtain

Max. *zq*

-2*zq +* 2*y2* ≤ 1 … (4.32)

3*zq* - 4*y2* ≤ 1 … (4.33)

-3*zq* + 5*y2* ≤ 1… (4.34)

5*zq* - 9*y2* ≤ 1  … (4.35)

-*zq* + *y2* ≤ 0 … (4.36)

- *y2* ≤ 0 … (4.37)

In the second stage of Gauss elimination technique with the help of (4.32), we have

Max. *zq*

*zq* ≥ -3 … (4.38)

*zq* ≤ 3 … (4.39)

*zq* ≥ … (4.40)

*zq* ≥ -1 … (4.41)

*zq* ≥ … (4.42)

Here, we get five bounded values of *zq*. i.e., {--5, }.

Since problem is maximization linear programming problem so, we will take *zq* = 3. Also, *zq* = 3 satisfies inequalities (4.38) to (4.42) altogether and by back substitution it gives *y2* = 2 from inequalities (4.32) to (4.37). Now, putting *zq* = 3 and *y2* = 2 in inequalities (4.25) to (4.31), we get different bounded values for variable *y1* out of them *y1*= 1 is the only value that satisfies all inequalities (4.25) to (4.31) altogether. Hence *y1* = 1.

So, *zq* = 3 implies that = 3 and *V* = .

Also, *y1* = implies that *q*1 = and *y2* = implies that *q*2 = .

Hence, strategies for player B () and value of game = .

**Formulation of LPP for player A:**

LPP for player A is the dual of LPP of player B. So, we have

Min. *zp = x1 + x2 + x3 + x4*

Subject to -2*x1* + 3*x2* - 3*x3 +* 5*x4* ≥ 1

-*x2* + 2*x3* -4*x4* ≥ 1

and *xi* ≥ 0; *i*=1,2,3,4

where *xi* = ;*i*=1,2,3,4 and *zp* = .

Again, we make standard form for apply Gauss elimination technique on above LPP. We have

Min. *zp*

*zp* - *x1* - *x2* - *x3* - *x4* ≥ 0 … (4.43)

-2*x1* + 3*x2* - 3*x3 +* 5*x4* ≥ 1 … (4.44)

-*x2* + 2*x3* -4*x4* ≥ 1 … (4.45)

*x1* ≥ 0 … (4.46)

*x2* ≥ 0 … (4.47)

*x3* ≥ 0 … (4.48)

*x4* ≥ 0… (4.49)

In the first stage of Gauss elimination with the help of (4.43), we have

Min. *zp*

-2*zp* + 5*x2* - *x3 +* 7*x4* ≥ 1 … (4.50)

-*x2* + 2*x3* -4*x4* ≥ 1 … (4.51)

*zp* - *x2* - *x3* - *x4* ≥ 0 … (4.52)

*x2* ≥ 0 … (4.53)

*x3* ≥ 0 … (4.54)

*x4* ≥ 0… (4.55)

In the second stage of Gauss elimination with the help of (4.50), we

Min. *zp*

-2*zp* + 9*x3* -13*x4* ≥ 6 … (4.56)

*zp* -3*x3* +3*x4* ≥ -1 … (4.57)

2*x3* -4*x4* ≥ 1 … (4.58)

*x3* ≥ 0 … (4.59)

*x4* ≥ 0… (4.60)

In the third stage of Gauss elimination with the help of (4.56), we have

Min. *zp*

*zp* -4*x4* ≥ 3 … (4.61)

2*zp* -6*x4* ≥ 1 … (4.62)

*zp* + 3*x4* ≥ -1 … (4.63)

*x4* ≥ 0… (4.64)

In the fourth stage of Gauss elimination with the help of (4.61), we have

Min. *zp*

*zp*≥ -7 … (4.65)

*zp*≥ … (4.66)

*zp* ≥3 … (4.67)

Here, we get three bounded values of *zp*, i.e., {-7, , 3}.

From which only *zp*= 3 is satisfies inequalities (4.65) to (4.67) and by back substitution it gives *x4*= 0 from inequalities (4.61) to (4.64).

Now, putting *zp* = 3 and *x4*= 0 in inequalities (4.56) to (4.60), we get different bounded values for variable *x3*out of this *x3*= is the only value that satisfies all the inequalities (4.56) to (4.60) altogether.

From inequalities (4.50) to (4.55) we find *x2*= and finally by putting values of *zp,* *x4,* *x3*, *x2* in inequalities (4.43) to (4.49). we get *x1*= 0.

So, *zp* = 3 implies that = 3 and *V* = .

Also, *x1* = implies that *p*1 = 0,

*x2* = implies that *p*2 = ,

*x3* = implies that *p*3 =

and *x4* = implies that *p*4 = 0.

So, strategies for player A (0,, ,0) and value of game = .

Hence, the optimal solution of game problem by Gauss elimination technique is: -

Value of game = .

Strategies for player A (0,, ,0).

Strategies for player B ().

**5. Conclusion and Future Scope :**

The proposed Gauss elimination technique (GET) is quite easy to apply on game theoretical problem. Elimination technique is a different approach for finding solution of the game problem. Gauss elimination technique is least time-consuming technique and it has very easy calculation because GET is based upon the solution of inequalities. With the help of numerical illustration, we have proved that this technique is applicable for all those types of game problem which contains m×n type payoff matrix. The calculation is very complicated in simplex method as compared to GET. In this chapter, we have applied Gauss elimination technique on game problems which contains a particular payoff matrix of defined game. In future, new researchers can extend this elimination techniques on different type of game problems.

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