Fuzzy Approximations of a Functional Equation within Digital Spatial Image Encryption Schemes

M. Arunkumar

Department of Mathematics

Kalaignar Karunanidhi Government Arts College, Tiruvanamalai – 606 603, Tamil Nadu, India.

e-mail: drarun4maths@gmail.com

M. Prabakaran

Department of Mathematics

Kalaignar Karunanidhi Government Arts College, Tiruvanamalai – 606 603, Tamil Nadu, India. e-mail: prabhakaran906@gmail.com

E. Sathya

Department of Mathematics

Kalaignar Karunanidhi Government Arts College, Tiruvanamalai – 606 603, Tamil Nadu, India.

e-mail: sathy24mathematics@gmail.com

V. Chandiran

Department of Mathematics

Kalaignar Karunanidhi Government Arts College, Tiruvanamalai – 606 603, Tamil Nadu, India.

e-mail: chandhiranphd@gmail.com

ABSTRACT

In this study, we comprehensively investigate and robustly prove the stability of a generalized additive functional equation in the framework of fuzzy Banach spaces, employing sophisticated direct and fixed point techniques. Furthermore, the paper delivers a powerful application of these theoretical results to cutting-edge digital spatial image cryptography systems, implemented and validated through MATLAB.

Keywords— Linear-type functional equation, Ulam–Hyers-based generalized stability paradigm, Fuzzy Banach space, Banach's Contraction principle.

#  INTRODUCTION

The investigation of stability phenomena in functional equations traces back to a fundamental question posed by Ulam [48], which concerned the stability of group homomorphisms under perturbation. This inquiry was affirmatively resolved by Hyers [24] within the setting of Banach spaces, thereby initiating the formal study of functional equation stability. Since then, the theory has witnessed substantial developments, with various researchers [3, 23, 39, 42] contributing notable generalizations and significant results that have enriched the literature.

 **In this context, particular emphasis has been placed on the rigorous analysis of both the solutions and the stability characteristics of additive functional equations** were discussed in [1, 5, 6, 14, 30, 34, 36, 38, 42, 49].

In this paper, authors presents a comprehensive examination of the stability properties of the following generalized additive functional equation

$P \left\{Z \left(Pϙ+n\right)+Z \left(Pϙ-n\right)\right\}+Z \left(ϙ+Pn\right)+Z \left(ϙ-Pn\right)=Z \left(ϙ+n\right)+Z\left(ϙ-n\right)+2P^{2} Z\left(ϙ\right)$ (1.10)

in Fuzzy Banach space by employing both the direct analytical technique and the fixed point framework. Additionally, the practical implementation and relevance of equation (1.10) are explored through its application to digital spatial image encryption methods, developed using MATLAB.

# STABILITY CONSIDERATIONS IN FUZZY BANACH SPACE: DIRECT METHOD

 This section, we employ the formal structure of fuzzy normed spaces, as presented in [17]-[18] and [32].

We begin the proof of the stability result in this section by assuming $J\_{1}$, $\left(J\_{1},N\right)$ and $\left(J\_{2},N^{'}\right)$ are linear space, fuzzy normed space and fuzzy Banach space respectively.

## **THEOREM: 3.1** Let $℧\in \left\{1.,-1\right\}$. Let $∂:D^{2}\rightarrow (0,\infty ]$ be a function with $0<\left(\frac{A}{P}\right)^{℧}<1$

$N^{'}\left(∂\left(P^{℧k}ϙ,0\right),l\right)\geq N^{'}\left(A^{℧k}∂\left(ϙ,0\right),l\right)$ (3.1)

for all $ϙ\in J\_{1}$ and all $A>0$ and

$\lim\_{k\to \infty }N^{'}\left(∂\left(A^{℧Z}ϙ,A^{℧Z}n\right),A^{℧Z}l\right)=1$ (3.2)

for all $ϙ,n\in J\_{1}$ and all $l>0$. Consider a mapping $Z:J\_{1} \rightarrow J\_{2}$ validating the inequality

$$N\left(P\left\{Z\left(Pϙ+n\right)+Z\left(Pϙ-n\right)\right\}+Z\left(ϙ+Pn\right)+Z\left(ϙ-Pn\right)-Z\left(ϙ+n\right)-Z\left(ϙ-n\right)-2P^{2}Z\left(ϙ\right),l\right)$$

$\geq N^{'}\left(∂\left(ϙ,n\right),l\right)$ (3.3)

for all $ϙ,n\in J\_{1}$ and all $l>0$.Then the limit,

$P\left(ϙ\right)=N-\lim\_{ k\to \infty }\frac{Z\left(P^{℧k}ϙ\right)}{P^{℧k}}$ (3.4)

exists for all $ϙ\in J\_{1}$ and the mapping $P:J\_{1} \rightarrow J\_{2}$ is a unique mapping satisfying (1.10) and

 $N\left(\left(P\left(ϙ\right)-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.2P\left|P-A\right|\right)$ (3.5)

for all $ϙ\in J\_{1}$ and all $l>0$.

## **PROOF:** For $℧$=1. Change $\left(ϙ,n\right)$ as $\left(ϙ,0\right)$ in (3.3) and using (F3), we get

$N\left(2PZ\left(Pϙ\right)-2P^{2}Z\left(ϙ\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l\right)$ (3.6)

$N\left(\left(\frac{Z\left(Pϙ\right)}{P}-Z\left(ϙ\right)\right),\frac{l}{2P^{2}}\right)\geq N^{'}\left(∂\left(ϙ,0\right),l\right)$ (3.7)

for all $ϙ\in J\_{1}$ and all $l>0$. Again, change $l$ as $l.2P^{2}$ in the above inequality, it gives

$N\left(\left(\frac{Z\left(Pϙ\right)}{P}-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.2P^{2}\right)$ (3.8)

for all $ϙ\in J\_{1}$ and all $l>0$.Substitute $ϙ$ as $Pϙ$ in (3.8) and using (F3), we achieve

 $N\left(\left(\frac{Z\left(P^{2}ϙ\right)}{P^{2}}-\frac{Z\left(Pϙ\right)}{P}\right),\frac{l}{P}\right)\geq N^{'}\left(∂\left(Pϙ,0\right),l.2P^{2}\right)$ (3.9)

for all $ϙ\in J\_{1}$ and all $l>0$. Again, substitute $ϙ$ as $J^{k}ϙ$ in (3.8) and using (F3), we receive

$N\left(\left(\frac{Z\left(P^{k+1}ϙ\right)}{P^{k+1}}-\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),\frac{l}{P^{k}}\right)\geq N^{'}\left(A^{k}∂\left(ϙ,0\right),l.2P^{2}\right)$ (3.10)

for all $ϙ\in J\_{1}$ and all $l>0$.Change $l$ as $l.P^{k}$ in the above inequality and using (F3), we have

$N\left(\left(\frac{Z\left(P^{k+1}ϙ\right)}{P^{k+1}}-\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.2P^{2}\left(\frac{P}{A}\right)^{k}\right)$ (3.11)

for all $ϙ\in J\_{1}$ and all $l>0$. It is easy show that,

 $\sum\_{i=0}^{k-1}\left(\frac{Z\left(P^{i+1}ϙ\right)}{P^{i+1}}-\frac{Z\left(P^{i}ϙ\right)}{P^{i}}\right)=\left(\frac{Z\left(P^{k}ϙ\right)}{P^{k}}-Z\left(ϙ\right)\right)$ (3.12)

for all $ϙ\in J\_{1}$ . Change $l$ as $\frac{l}{2P^{2}\left(\frac{P}{A}\right)^{k}}$ in (3.11), we obtain

$N\left(\left(\frac{Z\left(P^{k+1}ϙ\right)}{P^{k+1}}-\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),\frac{l}{2P^{2}\left(\frac{P}{A}\right)^{k}}\right)\geq N^{'}\left(∂\left(ϙ,0\right),l\right)$ (3.13)

for all $ϙ\in J\_{1}$ and all $l>0$. From (3.12) & (3.13)

$$N\left(\sum\_{i=o}^{k-1}\left(\left(\frac{Z\left(P^{i+1}ϙ\right)}{P^{i+1}}-\frac{Z\left(P^{i}ϙ\right)}{P^{i}}\right)\right),\frac{l}{2P^{2}}\sum\_{i=o}^{k-1}\left(\frac{A}{P}\right)^{i}\right)$$

 $\geq min\bigcup\_{i=0}^{k-1}\left\{N\left(\left(\frac{Z\left(P^{i+1}ϙ\right)}{P^{i+1}}-\frac{Z\left(P^{i}ϙ\right)}{P^{i}}\right),\frac{l}{2P^{2}}\left(\frac{A}{P}\right)^{i}\right)\right\}\geq N^{'}\left(∂\left(ϙ,0\right),l\right)$

which gives,

$N\left(\frac{Z\left(P^{k}ϙ\right)}{P^{k}}-Z\left(ϙ\right),\frac{l}{2P^{2}}\sum\_{i=o}^{k-1}\left(\frac{A}{P}\right)^{i}\right)$≥$N^{'}\left(∂\left(ϙ,0\right),l\right)$ (3.14)

for all $ϙ\in J\_{1}$ and all $l>0$. Change $ϙ$ as $P^{u}ϙ$ in above inequality, we get

$$N\left(\left(\frac{Z\left(P^{k+u}ϙ\right)}{P^{k+u}}-\frac{Z\left(P^{u}ϙ\right)}{P^{u}}\right),\frac{l}{2P^{2}}\sum\_{i=o}^{k-1}\left(\frac{A}{P}\right)^{i}\frac{1}{P^{u}}\right)\geq N^{'}\left(A^{u}∂\left(ϙ,0\right),l\right)$$

for all $ϙ\in J\_{1}$ and all $l>0$.Change $l$ as $A^{u}l$ in above inequality, we obtain

$N\left(\left(\frac{Z\left(P^{k+u}ϙ\right)}{P^{k+u}}-\frac{Z\left(P^{u}ϙ\right)}{P^{u}}\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),\frac{l}{\frac{l}{2P^{2}}\sum\_{i=o}^{k+u-1}\left(\frac{A}{P}\right)^{i}\left(\frac{A}{P}\right)^{u}}\right)$ (3.15)

for each $ϙ\in J\_{1}$ and every $l>0$ and $k,u\geq 0$. Since $0<A<P$ and $\sum\_{i=0}^{k}\left(\frac{A}{P}\right)^{i}$. By applying $N(x.)$ is a declining function on *R* with $\lim\_{t\to \infty }N\left(x,t\right)=1$ and Cauchy criterion convergence $\left\{\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right\}$ is a Cauchy sequence in $\left(J\_{2},N^{'}\right)$. Since $\left(J\_{2},N^{'}\right)$ is a fuzzy banach space. This sequence is converges to some point $P\left(ϙ\right)\in J\_{2}$. Let us define the function $P:J\_{1}\rightarrow J\_{2}$ by

$\lim\_{k\to \infty }N\left(\left(Z\left(ϙ\right)-\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),l\right)=1$ (3.16)

Let $u=0$ and $k\rightarrow \infty $ in (3.15)

$N\left(\left(P\left(ϙ\right)-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.P\left(P-A\right)\right)$ (3.17)

for all $ϙ\in J\_{1}$ and all $l>0$.

To prove that $P$ satisfies (1.10).Replace $ϙ$ as $P^{k} ϙ$ and $n$ as $P^{k}n$ in (3.3), we get

$$N\left(\left[P\left\{\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}\right\}+\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}-2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right],\frac{l}{P^{k}}\right)\geq N^{'}\left(∂\left(P^{k}ϙ ,P^{k}n \right),l\right)$$

for all $ϙ,n\in J\_{1}$ and all $l>0$.Change $l$ as $l.P^{k}$ in the above inequality, we obtain

$$N\left(\left[P\left\{\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}\right\}+\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}-2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right],l\right)$$

$\geq N^{'}\left(∂\left(P^{k}ϙ ,P^{k}n \right),l.P^{k}\right)$ (3.18)

$$N\left(P\left\{P\left(Pϙ+n\right)+P\left(Pϙ-n\right)\right\}+P\left(ϙ+Pn\right)+P\left(ϙ-Pn\right)-P\left(ϙ+n\right)-P\left(ϙ-n\right)-2P^{2}P\left(ϙ\right),l\right)$$

$$=N\left(P\left\{P\left(Pϙ+n\right)+P\left(Pϙ-n\right)\right\}+P\left(ϙ+Pn\right)+P\left(ϙ-Pn\right)-P\left(ϙ+n\right)-P\left(ϙ-n\right)-2P^{2}P\left(ϙ\right)+P\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}-P\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}+P\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}-P\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}+2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}}-2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}},l\right)$$

$$\geq N\left(P\left(P\left(Pϙ+n\right)-P\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}\right)+P\left(P\left(Pϙ-n\right)-P\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}\right)+\left(P\left(ϙ+Pn\right)-\frac{Z\left(P^{k}\left(+Pn\right)\right)}{P^{k}}\right)+\left(P\left(ϙ-Pn\right)-\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}\right)+\left(-P\left(ϙ+n\right)+\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}\right)+\left(-P\left(ϙ-n\right)+\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}\right)+\left(-2P^{2}P\left(ϙ\right)+2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right)+\left(P\left\{\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}\right\}+\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}-2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}+\frac{l}{8}\right)$$

$$\geq min\left\{N\left(P\left(P\left(Pϙ+n\right)-P\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}\right),\frac{l}{8}\right)+N\left(P\left(P\left(Pϙ-n\right)-P\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}\right),\frac{l}{8}\right)+N\left(P\left(ϙ+Pn\right)-\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}},\frac{l}{8}\right)+N\left(P\left(ϙ-Pn\right)-\frac{Z\left(P^{k}\left(ϙ-Jn\right)\right)}{P^{k}},\frac{l}{8}\right)+N\left(-P\left(ϙ+n\right)+\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}},\frac{l}{8}\right)+N\left(-P\left(ϙ-n\right)+\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}},\frac{l}{8}\right)+N\left(-2P^{2}P\left(ϙ\right)+2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}},\frac{l}{8}\right)+N\left(P\left\{\frac{Z\left(P^{k}\left(Pϙ+n\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(Pϙ-n\right)\right)}{P^{k}}\right\}+\frac{Z\left(P^{k}\left(ϙ+Pn\right)\right)}{P^{k}}+\frac{Z\left(P^{k}\left(ϙ-Pn\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ+n\right)\right)}{P^{k}}-\frac{Z\left(P^{k}\left(ϙ-n\right)\right)}{P^{k}}-2P^{2}\frac{Z\left(P^{k}ϙ\right)}{P^{k}},\frac{l}{8}\right)\right\}$$

(3.19)

From (3.18) & (3.19) and applying limit in (3.19), and also using (3.2) accordingly, we establish that

$$N\left(P\left\{P\left(Pϙ+n\right)+P\left(Pϙ-n\right)\right\}+P\left(ϙ+Pn\right)+P\left(ϙ-Pn\right)-P\left(ϙ+n\right)-P\left(ϙ-n\right)-2P^{2}P\left(ϙ\right),l\right)$$

$$\geq min\left\{1,1,1,1,1,1,1,1\right\}$$

which implies, *P* meets the condition of the equation (1.10).

In order to demonstrate the uniqueness of $P\left(ϙ\right)$, let us consider $P^{'}\left(ϙ\right)$ an alternative function that also satisfies the given functional equation (3.3) and (3.5)

$$N\left(\left(P\left(ϙ\right)-P^{'}\left(ϙ\right)\right),l\right)=N\left(\left(\frac{P\left(P^{k}ϙ\right)}{P^{k}}-\frac{P^{'}\left(P^{k}ϙ\right)}{P^{k}}\right),l\right)$$

$$ =N\left(\left(\frac{P\left(P^{k}ϙ\right)}{P^{k}}-\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right)+\left(\frac{P^{'}\left(P^{k}ϙ\right)}{P^{k}}+\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),\frac{l}{2}+\frac{l}{2}\right)$$

$ \geq min\left\{N\left(\left(\frac{P\left(P^{k}ϙ\right)}{P^{k}}-\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),\frac{l}{2}\right),\left(\left(-\frac{P^{'}\left(P^{k}ϙ\right)}{P^{k}}+\frac{Z\left(P^{k}ϙ\right)}{P^{k}}\right),\frac{l}{2}\right)\right\}$

 $\geq min\left\{N^{'}\left(∂\left(ϙ,0\right),\frac{2lP\left(P-A\right)}{2}\left(\frac{P}{A}\right)^{k}\right),N^{'}\left(∂\left(ϙ,0\right),\frac{2lP\left(P-A\right)}{2}\left(\frac{P}{A}\right)^{k}\right)\right\} $

$$ \geq N^{'}\left(∂\left(ϙ,0\right),\frac{2lP\left(P-A\right)}{2}\left(\frac{P}{A}\right)^{k}\right)$$

for all $ϙ\in J\_{1}$ and all $l>0$. Since $\lim\_{k\to \infty }\frac{2lP\left(P-A\right)}{2}\left(\frac{P}{A}\right)^{k}=\infty $, we obtain $\lim\_{k\to \infty }N^{'}\left(∂\left(ϙ,0\right),\frac{2lP\left(P-A\right)}{2}\left(\frac{P}{A}\right)^{k}\right)=1$,

so, $P\left(ϙ\right)=P^{'}\left(ϙ\right)$. Therefore, $P\left(ϙ\right)$ is unique. Thus, the theorem is proven for $℧=1$.

For $℧=-1$. Using (3.6), we get

$N\left(Z\left(Pϙ\right)-PZ\left(ϙ\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.2P\right)$ (3.20)

for all $ϙ\in J\_{1}$ and all $l>0$.Replace $ϙ$ as $\frac{ϙ}{P}$ in above inequality, we obtain

$N\left(Z\left(ϙ\right)-Z\left( \frac{ϙ}{P}\right),l\right)\geq N^{'}\left(∂\left( \frac{ϙ}{P},0\right),l.2P\right)$ (3.21)

for all $ϙ\in J\_{1}$ and all $l>0$. Change $ϙ $ as $\frac{ϙ}{P^{k+1}}$ in (3.14), we obtain

$N\left(P^{k}Z\left(\frac{ϙ}{P^{k}}\right)-P^{k+1}Z\left(\frac{ϙ}{P^{k+1}}\right),lP^{k}\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.2PA^{k+1}\right)$ (3.22)

for all $ϙ\in J\_{1}$ and all $l>0$. The rest of the reasoning is identical to that employed in the earlier scenario.

This completes the proof.

**Corollary 3.2** Suppose a function $Z:J\_{1}\rightarrow J\_{2}$ adheres to the inequality

$N\left(P\left\{Z\left(Pϙ+n\right)+Z\left(Pϙ-n\right)\right\}+Z\left(ϙ+Pn\right)+Z\left(ϙ-Pn\right)-Z\left(ϙ+n\right)-Z\left(ϙ-n\right)-2P^{2}Z\left(ϙ\right),l\right)$

 $\geq \left\{\begin{array}{c}τ\\τ\left\{\left|\left|v\right|\right|^{c}+\left|\left|w\right|\right|^{c}\right\}\end{array}\right.$ (3.23)

for all $ϙ,n\in J\_{1}$ and all $l>0$. where τ and c are constants with $τ>0$.Then there is a uniquely determined mapping $P:J\_{1}\rightarrow J\_{2}$ which adheres to the functional equation (3.5) and

$N\left(\left(Z\left(ϙ\right)-P\left(ϙ\right)\right),l\right)\geq \left\{\begin{array}{c}N^{'}\left(τ,l2P\left|P-1\right|\right), P\ne 1\\N^{'}\left(τ\left|\left|ϙ\right|\right|^{c},l2P\left|P-P^{c}\right|\right), P^{c}\ne P,c\ne 1\end{array}\right.$ (3.24)

for all $ϙ\in J\_{1}$ and all $l>0$.

# STABILITY CONSIDERATIONS IN FUZZY BANACH SPACE USING FIXED POINT PRINCIPLES

 This section is devoted to examining the stability of the generalized additive functional equation (1.10) in a fuzzy Banach space by means of the fixed point approach.

## **THEOREM: 4.1 [33]** Let $\left(X,d\right)$ be a complete generalized metric space and Let $P:X\rightarrow Y$ be a strictly contractive mapping with Lipschitz constant $L<1$.Then for each given element $x\in X$, either

$$d\left(P^{n}x,P^{n+1}x\right)=\infty $$

for all non negative integers n**.**

or there exists positive integers $n\_{0}$ such that

1. $d\left(P^{n}x,P^{n+1}x\right)<\infty $for all$n\geq n\_{0}$.
2. The sequence $\left\{P^{n}x\right\}$ converges to a fixed point $y^{\*}$ of $P$.
3. $y^{\*}$ is the unique fixed point of $P$ in the set $Y=\left\{y\in X/d\left(P^{n\_{0}}x,y<\infty \right)\right\}$.
4. $d\left(y,y^{\*}\right)\leq \left(\frac{1}{1-L}\right)d\left(y,Py\right)$ for all y ϵX.

## **THEOREM: 4.2** Let $Z:J\_{1} \rightarrow J\_{2}$ be a function where one can find a function $∂:D^{2}\rightarrow (0,\infty ]$ subject to the inequality

## $$N\left(P\left\{Z\left(Pϙ+n\right)+Z\left(Pϙ-n\right)\right\}+Z\left(ϙ+Pn\right)+Z\left(ϙ-Pn\right)-Z\left(ϙ+n\right)-Z\left(ϙ-n\right)-2P^{2}Z\left(ϙ\right),l\right)$$

$\geq N^{'}\left(∂\left(ϙ,n\right),l\right)$ (4.1)

with the condition

$\lim\_{k\to \infty }N^{'}\left(∂\left(F\_{r}^{k}ϙ,F\_{r}^{k}n\right),F\_{r}^{k}l\right)=1$ (4.2)

for every $ϙ,n\in J\_{1}$ and all $l>0$. If there exists $F=F\left(r\right)$ for which

$F\_{i}=\left\{\begin{array}{c}P, if r=0\\\frac{1}{P}, if r=1\end{array}\right.$ (4.3)

has the property

 $ϙ\rightarrow Z\left(ϙ\right)=\frac{1}{2P}∂\left(\frac{ϙ}{P},0\right)$ (4.4)

$N^{'}\left(L\frac{Z\left(F\_{r}ϙ\right)}{F\_{r}},l\right)=N^{'}\left(Z\left(ϙ\right),l\right)$ (4.5)

for each $ϙ\in J\_{1}$ and every $l>0$. The mapping $P:J\_{1} \rightarrow J\_{2}$ is the uniquely determined mapping fulfilling (1.10) and

 $N\left(\left(P\left(ϙ\right)-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(\frac{L^{1-i}}{1-L}Z\left(ϙ\right), l\right)$ (4.6)

for all $ϙ\in J\_{1} $and all $l>0$.

***PROOF:*** Let us examine the set $ ∇=\left\{Z| Z:J\_{1} \rightarrow J\_{2},p\left(0\right)=0\right\}$ and set forth the generalized metric on $∇$ by,

$d\left(Z,h\right)=inf\left\{ℇ\in \left(0,\infty \right) / N\left(Z\left(ϙ\right)-h\left(ϙ\right),l\right)\geq N^{'}\left(Z\left(ϙ\right),ℇl\right)\right\}$ (4.7)

It can be readily seen that $\left(∇,d\right)$ is complete. Delimit $D:∇\rightarrow ∇$ by

$D Z\left(ϙ\right)=\frac{Z\left(F\_{r}ϙ\right)}{F\_{r}}$ for all $ϙ\in B\_{1}$ (4.8)

Now for $Z,h\in ∇$ ,we have

$$d\left(Z.h\right)=ℇ$$

which implies $ N\left(Z\left(ϙ\right)-h\left(ϙ\right),l\right)\geq N^{'}\left(Z\left(ϙ\right),ℇl\right)$

$$N\left\{F\_{r}\left(\frac{Z\left(F\_{r}ϙ\right)}{F\_{r}}-\frac{h\left(F\_{r}ϙ\right)}{F\_{r}}\right),l\right\}\geq N^{'}\left(Z\left(F\_{r}ϙ\right),ℇl\right)$$

$$N\left\{\left(\frac{Z\left(F\_{r}ϙ\right)}{F\_{r}}-\frac{h\left(F\_{r}ϙ\right)}{F\_{r}}\right),\frac{l}{F\_{r}}\right\}\geq N^{'}\left(Z\left(F\_{r}ϙ\right),ℇl\right)$$

for all $ϙ\in J\_{1}$ and all $l>0$.Replace$ l$ as $F\_{r}l$ in above inequality, we get

$$N\left\{\left(D Z\left(ϙ\right)-D h\left(ϙ\right)\right),l\right\}\geq N^{'}\left(Z\left(F\_{r}ϙ\right),ℇF\_{r}l \right)$$

$$N\left\{\left(D Z\left(ϙ\right)-D h\left(ϙ\right)\right),l\right\}\geq ℇL$$

$d\left(DZ,Dh\right)\leq Ld\left(Z,h\right).$ (4.9)

for all $ϙ\in J\_{1}$ and all $l>0$. D is strictly contractive mapping on $∇$ with Lipschtiz constant $L$.

It follows from (3.7), that

 $N\left(\left(\frac{Z\left(Pϙ\right)}{P}-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(∂\left(ϙ,0\right),l.2P^{2}\right)$

$$N\left(\left(\frac{Z\left(Pϙ\right)}{P}-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(\frac{1}{2P}∂\left( ϙ,0\right),l.P\right)$$

$d\left(DZ,Z\right)\leq L=L^{1-r}$ (4.10)

for all $ϙ\in J\_{1}$ and all $l>0$.It follows from (3.15), that

 $ N\left(Z\left(ϙ\right)-PZ\left( \frac{ϙ}{P}\right),l\right)\geq N^{'}\left(\frac{1}{2P}∂\left( \frac{ϙ}{P},0\right),l\right)$

$$N\left(Z\left(ϙ\right)-PZ\left( \frac{ϙ}{P}\right),l\right)\geq N^{'}\left(Z\left(ϙ\right),l\right)$$

$d\left(Z,DZ\right)\leq 1=L^{1-r}$ (4.11)

for all $ϙ\in J\_{1}$ and all $l>0$.From (4.10) & (4.11), we conclude,

$ d\left(Z,DZ\right)\leq L^{1-r}<\infty $ (4.12)

which [FP1] holds. Invoking the fixed point property [FP2] in each instance, it gives that there exists a fixed point P of D in $∇$ such that

$P\left(ϙ\right)=N-\lim\_{ k\to \infty }\frac{Z\left(F\_{r}^{k}ϙ\right)}{F\_{r}^{k}}$ for all $ϙ\in B\_{1}$

Hence P obeys the functional equation (1.10). By [FP3], since $P$ is unique fixed point of D in the set

$$∀=\left\{Z\in ∇ | d\left(Z,P\right)<\infty \right\}$$

Therefore P is unique mapping such that $N\left(\left(Z\left(ϙ\right)-P\left(ϙ\right)\right),l\right)\geq N^{'}\left(Z\left(ϙ\right),ℇl\right)$

Finally by [FP4], we obtain

 $d\left(Z,P\right)\leq \frac{1}{1-L}d\left(Z,DZ\right)$

$$N\left(\left(P\left(ϙ\right)-Z\left(ϙ\right)\right),l\right)\geq N^{'}\left(\frac{L^{1-i}}{1-L}Z\left(ϙ\right), l\right)$$

for all $ϙ\in J\_{1}$ and all $l>0$. This concludes the proof of the theorem.

**Corollary 4.3:**

Suppose a function $Z:J\_{1}\rightarrow J\_{2}$ which adheres to the inequality

$$N\left(P\left\{Z\left(Pϙ+n\right)+Z\left(Pϙ-n\right)\right\}+Z\left(ϙ+Pn\right)+Z\left(ϙ-Pn\right)-Z\left(ϙ+n\right)-Z\left(ϙ-n\right)-2P^{2}Z\left(ϙ\right),l\right)$$

$\geq \left\{\begin{array}{c}τ\\τ\left\{\left|\left|v\right|\right|^{c}+\left|\left|w\right|\right|^{c}\right\}\end{array}\right.$ (4.13)

for all $ϙ,n\in J\_{1}$ and all $l>0$. where τ and c are constants with $τ>0$. One can assert the existence of a unique mapping$ P:J\_{1}\rightarrow J\_{2}$ which adheres to the functional equation (3.5) and

$N\left(\left(Z\left(ϙ\right)-P\left(ϙ\right)\right),l\right)\geq \left\{\begin{array}{c}N^{'}\left(τ,l2P\left|P-1\right|\right), P\ne 1\\N^{'}\left(τ\left|\left|ϙ\right|\right|^{c},l2P\left|P-P^{c}\right|\right), P^{c}\ne P,c\ne 1\end{array}\right.$ (4.14)

for all $ϙ\in J\_{1}$ and all $l>0$.

**VI. FUNCTIONAL EQUATIONS BASED SPATIAL IMAGE CRYPTO TECHNIQUE**

The term remote sensing takes on a specific implication dealing with space-borne imaging systems used to remotely sense the surface. Remote sensing is defined as data collected from a distance without visiting or interacting directly. When the distance between the object and viewer is large, or rather small, remote sensing approach suggests the use of spatial image. In modern days, the image based cryptographic techniques have advocated new and efficient ways to develop secure spatial image encryption techniques, see [2], [6].

In this research work, functional equations are used to improve the level of security in spatial image encryption. We apply functional equation (1.10) in digital spatial image crypto techniques system using MATLAB. An elementary idea is to encrypt the digital spatial image by applying the left hand side of (1.10). As the result, the intricate cypher image is obtained. See figures 6.1 and 6.2.

### Positioning Figures and Tables: Place figures and tables at the top and bottom of columns. Avoid placing them in the middle of columns. Large figures and tables may span across both columns. Figure captions should be below the figures; table heads should appear above the tables. Insert figures and tables after they are cited in the text. Make them bold (figure and table title).



***Figure 6.1. Encryption***



***Figure 6.2. Image Encryption***

When cypher image reaches the receiver, he must use right hand side of (1.1) as a key. On entering the accurate key, the MATLAB code decrypts the entire image and provides original image to the receiver. See

figures 6.3 and 6.4.



***Figure 6.3. Decryption***



***Figure 6.4. Image Decryption***

4.1. Security Analysis. The distinctive approach in applying functional equations on spatial image crypto technique is, we use two different keys with same solutions that are LHS of functional equations for encrypting and RHS of functional equations for decrypting, whereas, traditional systems like DES, Triple- DES, RSA and IDEA use single key for both encryption and decryption. This uniqueness of functional equation progresses the security level of transmitting spatial image and overwhelmed traditional techniques limitations. A statistical analysis shows that the tactic for image crypto technique provides an effective and secure way for real time spatial image encryption and transmission from the cryptographic viewpoint.

# CONCLUSION

We introduced a generalized additive functional equation, obtained its general solution and stabilities in modular space by using fixed point theory. Also, we applied (1.10) in digital spatial image crypto techniques system using MATLAB.

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