**Modeling of Modified Fourier Elimination Technique for Game Theory**

**Sanjay Jain1, Adarsh Mangal2, Priyanka Jain3\***

1Principal, Government College Nand, Ajmer, India

2Department of Mathematics, Engineering College Ajmer, Ajmer, India

3\*Department of Mathematics, Research Scholar of Mathematics, S. P. C. Government College Ajmer, Ajmer, India

drjainsanjay@gmail.com

dradarshmangal1@gmail.com

93priyankajain@gmail.com

**ABSTRACT**

In this chapter, a new approach named Modified Fourier Elimination Technique (MFET) is proposed to find the optimal strategies and the value of the games of any game theoretical problem. This technique is based upon the concept of bounds. Before apply the suggested MFET, first of all we have to form two Linear Programming Problems (LPP) for both the row and column players by the given game problem containing m rows and n columns. By applying the MFET separately to both the linear programming problems, one can get the optimal solution of both the linear problems and as well game problem. The proposed technique is easy to understand as compared to Simplex Method for linear programming problems in terms of computational complexity.

**Keywords:** Optimal solution,Game theoretical method, Linear programming problem, Modified Fourier Elimination Technique.

**1. INTRODUCTION**

Game theory is an important tool of operation research. Game theoretical models help us to convert any physical situation into a mathematical problem. Mathematical modeling of the real-world problem gives optimize solution with accuracy. There exist so many methods to solve game theoretical problem.

In this chapter, we will solve game problem by an elimination technique named modified Fourier elimination technique (MFET). This technique is used in numerical analysis for solve system of linear equations. Modified Fourier elimination technique has already used to solve different type of mathematical programming problems such as linear programming problems, fractional programming problems, quadratic programming problem, multi-objective programming problem, etc. This technique is very effective in the field of numerical analysis.

**2. Modified Fourier Elimination Technique for Game Problem:**

In this chapter, we apply modified Fourier elimination technique on game theoretical problems. for this, first we formulate linear programming problem form of game problem for one player. LPP form of any game problem contains objective function, constraints and non-negative restriction of variables. Since constraints and non-negative restrictions are inequalities, so that for apply modified Fourier elimination technique on LPP of game problem we convert this LPP in standard form by treating objective function as constraint. Now, we have a system of linear inequalities. When we apply MFET for system of inequalities, the required necessary condition is that all the inequalities are of same sign. It means all inequalities should be either greater than equal to (≥) or less than equal to (≤) in nature. Therefore, we make same sign of all inequalities. After that we construct following pairwise disjoint sets , and for each variable.

 = the set of all constraints in which coefficient of *jth* variable is positive.

 = the set of all constraints in which coefficient of *jth* variable is negative.

 = the set of all constraints in which coefficient of *jth* variable is zero.

For selection of eliminating variable, we calculate the value of following equation for each variable.

Here, denote the number of constraints present in the set . From the above relation we get index *j* corresponding to minimum, select this variable of index *j* for elimination. Now for elimination of , we will combine inequalities of with inequalities of . Repeating this process for (n-1) times, we are left with only one variable. Finally, we get the value of other variables by back substitution method. We calculate the value of game and optimal strategies of one player with the help of the values of variables.

Now, by the concept of duality we formulate the LPP form of game problem for another player and repeat the same process to find out the value of game and optimal strategies for second player.

**3. Problem Formulation for Modified Fourier Elimination Technique**

The general form of game problem can be written as:

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In this chapter, we will apply modified Fourier elimination technique to calculate probabilities to choose strategies for both players A and B for best outcomes. Also, we will calculate the value of game. Now, we formulate the game problem for modified Fourier elimination technique. For this, first of all we write the linear programming problem form of the game problem for each player.

Let *V* be the value of game or expected gain or lose. Here, we consider that Player B’s objective is to minimize the expected loss, which can be achieved by minimizing V or maximize .

So, the expected losses for player B will be as follows:

Dividing the above constraints by V, we get

 + + + 1

 + + + 1

 + + + 1

For simplification, take = ; j= 1, 2…...n.

In order to minimize V, player B can maximize .

Since

Divide above equation by V. We get,

 + + + =

Let = So, = = .

**LPP form of game problem for player B:**

 Max. = =

 Subject to

 and ≥ 0; j = 1, 2…...n.

 where = ; j= 1, 2…...n.

**LPP form of game problem for player A:**

Min. = =

 Subject to

 and ; i = 1, 2……m.

 where = ; i = 1, 2……m.

Now for solution of game problem by modified Fourier elimination technique, we have to convert the LPP form of game problem into a standard form. The standard form for modified Fourier elimination technique contains either equations or inequalities with same sign. So, we convert objective function into inequality. Also, ensure that sign of inequalities is same for converted objective function, constraints and restricted variables.

**Standard form of game problem for modified Fourier elimination technique:**

**For player B**

Max.

 ; j = 1, 2……, n.

 where, = ; j= 1, 2…...n.

**For player A**

Min.

 ; i = 1, 2……m.

where, = ; i = 1, 2……m.

We will apply modified Fourier elimination technique on above standard forms one by one.

**The Algorithm:**

The algorithm to solve the game problem is same for each player. We will follow following steps:

**Step – 1** Let us construct the following pairwise disjoint sets for all existing variables.

 = {*i*: 0}

 = {*i*: 0}

 = {*i*: 0};

*i* = 1, 2…., m and *j* = 1, 2…., n.

**Step – 2** The optimal solution of game problem exists when neither absurd inequality nor empty set of or exist in the system.

**Step – 3** For selection of eliminating variable, we calculate

Where denote the number of constraints present in the set S.

**Step – 4** Choose the index j corresponding to the minimum that calculated in the previous step.

**Step – 5** For elimination of the variable that chosen in the step-4, combine each inequality of with every inequality of .

**Step – 6** Repeat the above 5 steps until the entire decision variables are not eliminated.

To explain the entire procedure of modified Fourier elimination technique, we will solve some examples of game problem.

**4. Numerical Illustration**

**1.** Solve the following game problem:

 Player B

 PlayerA

Now let us assume that *p1* & *p2*are probabilities for player A and *q1*& *q2* are probabilities for player B to choose strategies for maximum gain of player A and minimum loss of player B.

**Formulation of LPP for player B:**

Let *V* be the maximum gain of player A. Then for minimum loss of player B we have to minimize *V*. As we formulated the generalized LPP form of game problem, we can directly write the LPP for player B for this example. i.e.,

 Max. *zq =*   *+*

Subject to 6 + 9 ≤ 1

 8+ 4≤ 1

 and *,*≥ 0

Where, = and = .

Also, min. *V* = max. = max. *zq*.

Now convert the above LPP in standard form for apply MFET, we get

 Max. *zq*

*zq - - ≤* 0… (4.1)

6 *+* 9 *≤* 1… (4.2)

8 *+* 4 *≤* 1… (4.3)

*- ≤* 0… (4.4)

 *-≤* 0 …(4.5)

In the first stage of modified fourier elimination technique, we construct pairwise disjoint sets of inequalities on the basis of their variables sign.

For variable = {(4.2), (4.3)} = Two constraints

 = {(4.1), (4.4)} = Two constraints

 = {(4.5)} = One constraint

for variable = {(4.2), (4.3)} = Two constraints

 = {(4.1), (4.5)} = Two constraints

 = {(4.4)} = One constraint

Then,

 =

 = min {2\*2, 2\*2} = min {4, 4} = 4 for *j =* 1 or 2

Hence, *j* = 1or 2 implies that any of or  can select for elimination. Let us eliminate . For this we combine each inequality of with each inequality of . We get,

 Max*. zq*

6*zq* + 3 ≤ 1 … (4.6)

 9 ≤ 1 … (4.7)

 8*zq* - 4≤ 1 … (4.8)

 4≤ 1 … (4.9)

 *-* ≤ 0 … (4.10)

Again, we construct , and for elimination of .

 = {(4.6), (4.7), (4.9)} = Three constraints

 = {(4.8), (4.10)} = Two constraints

 = { } = empty set

Now combine inequalities of with inequalities of . We get,

 Max. *zq*

 *zq* ≤  … (4.11)

 *zq* ≤ … (4.12)

 *zq* ≤ … (4.13)

 *zq* ≤ … (4.14)

Here, we get four bounded values of *zq*. From these four values only *zq* = satisfies inequalities (4.6) to (4.10) altogether. So, taking *zq* = . We find = . Now, putting *zq* = and = in inequalities (4.1) to (4.5), we find = .

So, *zq* = implies that = and *V* =

Also, = implies that  = and = implies that  = .

Hence, strategies for player B (, ) and value of game = .

**Formulation of LPP for player A:**

LPP for player A is dual of LPP of player B. So, we have

 Min. *zp = +*

Subject to 6 + 8 ≥ 1

 9+ 4 ≥ 1

 and *,*≥ 0

where = and  = .

Again, we make standard form for apply MFET on above LPP. We have

 Min. *zp*

*zp - -*  ≥ 0 … (4.15)

 6 *+* 8≥ 1 … (4.16)

 9+ 4≥ 1 … (4.17)

 ≥ 0 … (4.18)

 ≥ 0… (4.19)

Now we construct pairwise disjoint sets of inequalities on the basis of their variables sign.

For variable = {(4.16), (4.17), (4.18)} = Three constraints

 = {(4.15)} = One constraint

 = {(4.19)} = One constraint

for variable  = {(4.16), (4.17), (4.19)} = Three constraints

 = {(4.15)} = One constraint

 = {(4.18)} = One constraint

Then,

 =

 = min {3\*1, 3\*1} = min {3,3} = 3 for *j =* 1 or 2

Hence, *j* = 1or 2 implies that any of or  can select for elimination. Let us eliminate . For this we combine each inequality of with each inequality of . We get,

 Min. *zp*

 6*zp* + 2 ≥ 1 … (4.20)

 9*zp* – 5 ≥ 1 … (4.21)

 *zp* - ≥ 0 … (4.22)

 ≥ 0 … (4.23)

Again, we construct , and for elimination of *x2*.

 = {(4.20), (4.23)} = Two constraints

 = {(4.21), (4.22)} = Two constraints

 = { } = empty set

Now combine inequalities of with inequalities of . We get,

 Min. *zq*

 *zp* ≥  … (4.24)

 *zp ≥* … (4.25)

 *zp* ≥ … (4.26)

 *zp ≥* 0 … (4.27)

Here, we get four bounded values of *zp*. From these only *zp* = satisfies inequalities (4.20) to (4.23) altogether. So, taking *zp* = . We find = . Now, putting *zp* = and  = in inequalities (4.15) to (4.19), we find = .

So, *zp* = implies that = and *V* =

Also, = implies that = and = implies that = .

Hence, strategies for player A (, ) and value of game = .

So, by Modified Fourier Elimination Technique (MFET) the optimal solution of game problem is:

value of game = .

strategies for player A (, ).

strategies for player B (, ).

**2.** Solve the following game problem:

 Player B

 PlayerA

Now let us assume that *p1* & *p2*are probabilities for player A and *q1*, *q2,* *q3, q4* & *q5* are probabilities for player B to choose strategies for maximum gain of player A and minimum loss of player B.

**Formulation of LPP for player B:**

Let *V* be the maximum gain of player A. Then for minimum loss of player B we have to minimize *V*. As we formulated the generalized LPP form of game problem, we can directly write the LPP for player B for this example. i.e.,

 Min. *V*

 Subject to  *V*

 *+* 5 *V*

 *+* + *=* 1

Divide above inequalities and equation by V. Then,

Let *yj* = ; j=1, 2,……..,5

Also, min. *V* = max. = max. *zq.* So,

 Max. *zq =*

Subject to 3 ≤ 1

 ≤ 1

 and *,*≥ 0

Now convert the above LPP in standard form for apply MFET, we get

 Max. *zq*

*zq - - - - - ≤* 0… (4.28)

 *+* 6 - *+* 7*≤* 1… (4.29)

- *+* 5 - 2 *≤* 1… (4.30)

*- ≤* 0… (4.31)

 *-≤* 0 … (4.32)

 *- ≤* 0 … (4.33)

 - *≤* 0 … (4.34)

 - *≤* 0 … (4.35)

In the first stage of modified Fourier elimination technique, we construct pairwise disjoint sets of inequalities on the basis of their variables sign.

For variable , = {(4.29)} = One constraint

 = {(4.28), (4.30), (4.31)} = Three constraints

 = {(4.32), (4.33), (4.34), (4.35)} = Four constraints

for variable, = {(4.30)} = One constraint

 = {(4.28), (4.32)} = Two constraints

 = {(4.29), (4.31), (4.33), (4.34), (4.35)} = five constraints

for variable, = {(4.29)} = One constraint

 = {(4.28), (4.30), (4.33)} = Three constraints

 = {(4.31), (4.32), (4.34), (4.35)} = Four constraints

for variable,  = {(4.30)} = One constraint

 = {(4.28), (4.29), (4.34)} = Three constraints

 = {(4.31), (4.31), (4.33), (4.35)} = Four constraints

for variable , = {(4.29), (4.30)} = Two constraint

 = {(4.28), (4.35)} = Two constraints

 = {(4.31), (4.32), (4.33), (4.34)} = Four constraints

Then, =

 = min {1\*3,1\*2,1\*3,1\*3, 2\*2}

 = min {3, 2, 3, 3, 4} = 2

Here, = 2, for *j* = 2 this implies that will be eliminate. For this we will combine each inequality of with each inequality of . We get,

 Max*. zq*

5*zq* - 6 - 7 - 3 - 4 ≤ 1 … (4.36)

 - - 2 + 2 + ≤ 1 … (4.37)

 3+ 6 - + 7≤ 1 … (4.38)

 *-*≤ 0 … (4.39)

 *- ≤* 0 … (4.40)

 - 0 … (4.41)

 - 0 … (4.42)

Again, we will construct pairwise disjoint sets of inequalities for elimination of next variable.

So, for variable , = {(4.38)} = One constraint

 = {(4.36), (4.37), (4.39)} = Three constraints

 = {(4.40), (4.41), (4.42)} = Three constraints

for variable , = {(4.38)} = One constraints

 = {(4.36), (4.37), (4.40)} = Three constraints

 = {(4.39), (4.41), (4.42)} = Three constraints

for variable , = {(4.37)} = One constraint

 = {(4.36), (4.38), (4.41)} = Three constraints

 = {(4.39), (4.40), (4.42)} = Three constraints

for variable , = {(4.37), (4.38)} = Two constraints

 = {(4.36), (4.42)} = Two constraints

 = {(4.39), (4.40), (4.41)} = Three constraints

Then, =

 = min {1\*3,1\*3,1\*3, 2\*2}

 = min {3, 3, 3, 4} = 3

Here, = 3, for *j* = 1,3or 4 this implies that any of ,or can select for elimination. Let us eliminate . For this we will combine each inequality of with each inequality of . We get,

 Max*. zq*

5*zq* + 5- 5 + 10 3 … (4.43)

 5 + 10 4 … (4.44)

 6 - + 7 1 … (4.45)

 - 0 … (4.46)

 - 0 … (4.47)

 - 0 … (4.48)

Again, for remaining variables, we will construct pairwise disjoint sets; i.e.,

For variable , = {(4.43), (4.45)} = Two constraints

 = {(4.46)} = One constraint

 = {(4.44), (4.47), (4.48)} = Three constraints

for variable , = {(4.44)} = One constraint

 = {(4.43), (4.45), (4.47)} = Three constraints

 = {(4.46), (4.48)} = Two constraints

for variable , = {(4.43), (4.44), (4.45)} = Three constraints

 = {(4.48)} = One constraint

 = {(4.46), (4.47)} = Two constraints

Then, =

 = min {2\*1,1\*3, 3\*1} = min {2, 3, 3} = 2

Here, = 2, for *j* = 3 this implies thatwill be select for elimination. For this we will combine each inequality of with each inequality of . We get,

 Max*. zq*

 5*zq*- 5 + 10 3 … (4.49)

 - + 7 1 … (4.50)

 5 + 10 4 … (4.51)

 - 0 … (4.52)

 - 0 … (4.53)

Now, for selection of eliminating variable from and , we will construct pair-wise disjoint sets of inequalities. i.e.,

For variable , = {(4.51)} = One constraint

 = {(4.49), (4.50), (4.52)} = Three constraints

 = {(4.53)} = One constraint

for variable , = {(4.49), (4.50), (4.51)} = Three constraints

 = {(4.53)} = One constraint

 = {(4.52)} = One constraint

Then, =

 = min {1\*3, 3\*1}

 = min {3, 3} = 3

Here, = 3, for *j* = 4 or 5 this implies that any ofor can be select for elimination. Let us eliminate . For this we will combine each inequality of with each inequality of . We get,

 Max*. zq*

 5*zq*+ 20 7 … (4.54)

 … (4.55)

 … (4.56)

 - 0 … (4.57)

For elimination of , we have to construct and . So,

 = {(4.54), (4.55), (4.56)} = Three constraints

 = {(4.57)} = One constraint

 = { } = empty set

After combination of and we get,

 Max*. zq*

 *zq* ≤ … (4.58)

 ≥ 0 … (4.59)

 ≥ 0 … (4.60)

Hence, *zq* = implies that = and *V* = .

Now, by back substitution method we will get values of all variables as

 = 0, = , = 0, = and = 0

Since, *yj* = ; j = 1, 2……, n this implies that = *yj .*

So, *q1* = , *q2* = 0, *q3* =0, *q4* = and *q5* = 0.

Hence, the strategies for player B (,0) and value of game = .

**Formulation of LPP for player A:**

LPP for player A is dual of LPP of player B. So, we have

 Min. *zp = +*

Subject to 3 - ≥ 1

 5 ≥ 1

 6 - 2 ≥ 1

 - + 2 ≥ 1

 7 *+* ≥ 1

and *,*≥ 0

where = and  = .

Again, we will make standard form for apply modified Fourier elimination technique on above LPP. We have

 Min. *zp*

*zp - -*  ≥ 0 … (4.61)

  *-* ≥ 1 … (4.62)

 5≥ 1 … (4.63)

  *-* 2≥ 1 … (4.64)

 -+ 2 ≥ 1 … (4.65)

 7 +≥ 1… (4.66)

 ≥ 0 … (4.67)

 ≥ 0… (4.68)

Now we will construct pairwise disjoint sets of inequalities on the basis of their variables sign.

For variable, = {(4.62), (4.64), (4.66), (4.67)} = Four constraints

 = {(4.61), (4.65)} = Two constraints

 = {(4.63), (4.68)} = Two constraints

for variable , = {(4.63), (4.65), (4.66), (4.68)} = Four constraints

 = {(4.61), (4.62), (4.64)} = Three constraints

 = {(4.67)} = One constraint

Then, =

 = min {4\*2, 4\*3} = min {8,12} = 8

Here, = 8, for *j* = 1 this implies thatwill be select for elimination. For this we will combine each inequality of with each inequality of . We get,

 Min. *zp*

 3*zp* - 4 ≥ 1 … (4.69)

  5 ≥ 4 … (4.70)

 6*zp* - 8≥ 1 … (4.71)

 10≥ 7 … (4.72)

 7*zp* - 6≥ 1 … (4.73)

 15 ≥ 8 … (4.74)

 *zp* - ≥ 0 … (4.75)

 2 ≥ 1 … (4.76)

Again, we construct , and for elimination of *x2*.

 = {(4.70), (4.72), (4.74), (4.76)} = Four constraints

 = {(4.69), (4.71), (4.73), (4.75)} = Four constraints

 = { } = empty set

After combining inequalities of with inequalities of . We will get different bounded values of *zp*. From which only *zp* = is satisfies each inequality of previous system and gives = .

Now, putting *zp* = and  = in inequalities (4.34) to (4.41), we will find = .

So, *zp* = implies that = and *V* =

Also, = implies that = and = implies that = .

Hence, the strategies for player A (, ) and value of game = .

Hence by Modified Fourier Elimination Technique (MFET) the optimal solution of cited game problem is:

Value of game = .

Strategies for player A (, ).

Strategies for player B (,0).

**5. Conclusion and Future Scope :**

Modified Fourier elimination technique is an interesting technique to solve game theoretical problems. Construction of pairwise disjoint sets for each unknown variable is a very important part of this technique which help us to select eliminating variable. Since we select eliminating variable with the help of , therefore our calculation for elimination has reduced for that selected variable. In modified Fourier elimination technique, we are combining inequalities of the sets and which make the calculation very easy because they contain inequalities with opposite sign for the particular variable. This is new and different approach to solve that type of game problem which is either contains m × n payoff matrix or may be converted into m × n payoff matrix with the help of given data. In this chapter with the help of examples, we have explained two type of game problems i.e., 2×2 and 2×n. In future, we recommend new researchers to extend modified Fourier elimination techniques for big game problems.

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