**Modeling of Fourier Motzkin Elimination Technique for Game Theory**

**Sanjay Jain1, Adarsh Mangal2, Priyanka Jain3\***

1Principal, Government College Nand, Ajmer, India

2Department of Mathematics, Engineering College Ajmer, Ajmer, India

3\*Department of Mathematics, Research Scholar of Mathematics, S. P. C. Government College Ajmer, Ajmer, India

drjainsanjay@gmail.com

dradarshmangal1@gmail.com

93priyankajain@gmail.com

**ABSTRACT**

In this chapter, a different approach named Fourier Motzkin Elimination Technique (FMET) is proposed to find out the solution of game problem and optimal strategies for it. This technique is based upon the concept of bounds. In this technique, we have to form two linear programming problem (LPP) for both players by the given game problem having m×n payoff matrix. By applying the FMET separately to both the linear programming problems, one can get the optimal solution of both linear problems and as well game problem. This technique is uncomplicated to understand as compared to simplex method for solving game problems. Numerical examples are also given in this chapter to understand this elimination technique.

**Keywords:** Optimal strategies, Game problems, Linear programming problem, Fourier Motzkin Elimination Technique.

**1. INTRODUCTION**

In our daily life we make decisions even without noticing them. In simple situation decisions are taken simply by common sense, perception, experience and skills without using any mathematical or statistical model. Game theory deals with interactive situations where two or more than two players make decisions that jointly determine the final outcome. The gain of one player is equal to lose of another player. Hence, they are in conflict. Today, various methods are available for solution of different type of game problems such as graphical method, dominance rule, linear programming method, algebraic method etc.

In this chapter, a novel approach named Fourier Motzkin Elimination Technique (FMET) is discussed to find out the solution of game and optimal strategies for it. This technique is based upon the concept of bounds. In this technique, we have to form two linear programming problem (LPP) for both players by the given game problem.

**2. Fourier Motzkin Elimination Technique for Game Problem:**

We know that by solving equations one can get a single solution or value but when inequalities have solved then we find out more than one possibility for solution in bounded form. From these bounded values, we will select minimum value or maximum value according to the objective function of given problem. Our main point is to implement Fourier Motzkin Elimination Technique on system of inequalities instead of equations.

In modified Fourier elimination technique, we construct pairwise disjoint sets for all existing variables and calculate for selection of eliminating variable. While in Fourier Motzkin elimination technique we randomly select a variable for elimination and eliminate that by combine inequalities of and .

To apply Fourier Motzkin Elimination Technique on game problem, first we have to construct the following pairwise disjoint sets for all existing variables.

 = {i: 0}

 = {i: 0}

 = {i: 0}; i = 1, 2…., m and j = 1, 2…., n.

For, optimal solution the set or should not be empty. If anyone from these two sets becomes empty, then the problem has unbounded solution.

Now, we must eliminate the variables to achieve the optimality of the problem. We can eliminate variables by combining the inequalities of setand . In each iteration one variable has eliminated and at the end of the process there remains a single variable which have more than one bounded value. From all bounded values of this remaining variable, we can find the acceptable value of that last remaining variable.

Now from the process of back-substitution we can obtain the value of other variables and thus the optimal solution is reached.

**3. Problem Formulation for Fourier Motzkin Elimination Technique**

The general form of game problem can be written as:

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In this chapter, we are going to apply Fourier Motzkin elimination technique to obtain probabilities to choose strategies for both players A and B. Also, we will find out the value of game. Now, we formulate the game problem for Fourier Motzkin elimination technique. Let *V* be the value of game or well said expected gain or loss.

Here, we consider that Player B’s objective is to minimize the expected loss, which can be achieved by minimizing *V* because *V* is the loss of player B. The expected losses for player B can be written in the form of linear combination with the help of payoff matrix.

So, the expected losses for player B will be as follows:

Dividing the above constraints by V, we get

 + + + 1

 + + + 1

 + + + 1

For simplification, take = ; j= 1, 2…...n.

In order to minimize V, player B can maximize .

Since

Divide above equation by V. We get,

 + + + =

Let = So, = = .

**LPP form of game problem for player B:**

 Max. = =

 Subject to

 and ≥ 0; j = 1, 2…...n.

 where = ; j= 1, 2…...n.

Now, player A’s objective is to maximize the expected gains, which can be achieved by maximizing *V*, i.e., it might gain more than *V* if player B adopts a poor strategy.

**LPP form of game problem for player A:**

Min. = =

 Subject to

 and ; i = 1, 2……m.

 where = ; i = 1, 2……m.

Now for find out the solution of game problem by Fourier Motzkin elimination technique, we have to convert the LPP form of game problem into a standard form. The standard form for Fourier Motzkin elimination technique contains either equations or inequalities with same sign. So, we convert objective function into inequality. Also, ensure that sign of inequalities is same for converted objective function, constraints and restricted variables.

**Standard form of game problem for modified Fourier elimination technique:**

**For player B**

Max.

 ; j = 1, 2……, n.

 where, = ; j= 1, 2…...n.

**For player A**

Min.

 ; i = 1, 2……m.

where, = ; i = 1, 2……m.

We will apply Fourier Motzkin elimination technique on above standard forms one by one. The algorithm can be understood by following examples.

**4. Numerical Illustration**

**1.** Solve the following game problem:

 Player B

 PlayerA

Now let us assume that *p1* & *p2*are probabilities for player A and *q1*& *q2* are probabilities for player B to choose strategies for maximum gain of player A and minimum loss of player B.

**Formulation of LPP for player B:**

Let *V* be the maximum gain of player A. Then for minimum loss of player B we have to minimize *V*. So, LPP for player B can be written as

 Max. *zq =*   *+*

Subject to 5 + 2 ≤ 1

 3+ 4≤ 1

 and *,*≥ 0

 Where, *=*  and *=* and min. *V* = max. = max. *zq.*

Now convert the above LPP in standard form for apply Fourier Motzkin elimination technique, we get

 Max. *zq*

*zq - - ≤* 0… (4.1)

5 *+* 2 *≤* 1… (4.2)

3 *+* 4 *≤* 1… (4.3)

*- ≤* 0… (4.4)

 *-≤* 0 (4.5)

In the first stage of Fourier Motzkin elimination technique, we construct pairwise disjoint sets of inequalities for variable which is based on their variables sign. So,

 = {(4.2), (4.3)} = Two constraints

 = {(4.1), (4.5)} = Two constraints

 = {(4.4)} = One constraint

For elimination of y2 , combine inequalities of with ,we get

 Max*. zq*

2*zq* + 3 ≤ 1 … (4.6)

 5 ≤ 1 … (4.7)

 4*zq* - ≤ 1 … (4.8)

 3≤ 1 … (4.9)

Again, we construct , and for elimination of .

 = {(4.6), (4.7), (4.9)} = Three constraints

 = {(4.8)} = One constraints

 = { } = empty set

Now combine inequalities of with inequalities of for elimination of y1 . We get,

 Max. *zq*

 *zq* ≤  … (4.10)

 *zq* ≤ … (4.11)

 *zq* ≤ … (4.12)

 Here, we get three bounded values of *zq*. From these three values only *zq* = satisfies inequalities (4.6) to (4.9) altogether. So, taking *zq* = . We find = .

Now, putting *zq* = and = in inequalities (4.1) to (4.5), we find distinct bounded values for *y2*. Inequalities (4.1) to (4.5) are satisfied altogether by only one value from these distinct values i.e., = . Hence = .

So, *zq* = implies that = and *V* =

Also, = implies that  =

and = implies that  = .

Hence, strategies for player B (, ) and value of game = .

**Formulation of LPP for player A:**

LPP for player A is dual of LPP of player B. So, we have

 Min. *zp = +*

Subject to 5 + 3 ≥ 1

 2+ 4 ≥ 1

 and *,*≥ 0

where = and  = .

Again, we make standard form for apply fourier motzkin elimination technique on above LPP. We have

 Min. *zp*

*zp - -*  ≥ 0 … (4.13)

 5 *+* 3≥ 1 … (4.14)

 2+ 4≥ 1 … (4.15)

 ≥ 0 … (4.16)

 ≥ 0… (4.17)

Now we construct pairwise disjoint sets of inequalities for randomly selected variable on the basis of their coefficient sign.

 = {(4.14), (4.15), (4.17)} = Three constraints

 = {(4.13)} = One constraint

 = {(4.16)} = One constraint

combine inequalities of with for elimination of we get

 Min. *zp*

 3*zp* + 2 ≥ 1 … (4.18)

 4*zp* – 2 ≥ 1 … (4.19)

 *zp* - ≥ 0 … (4.20)

Again, we construct , and for elimination of *x1*.

 = {(4.18)} = One constraints

 = {(4.19), (4.20)} = Two constraints

 = { } = empty set

Now combine inequalities of with inequalities of . We get,

 Min. *zq*

 *zp* ≥  … (4.21)

 *zp ≥* … (4.22)

Here, we get two bounded values of *zp*. From these two values only *zp* = satisfies inequalities (4.18) to (4.20) altogether. So, taking *zp* = . We find = .Now, putting *zp* = and  = in inequalities (4.13) to (4.17), we find different bounded values for . Inequalities (4.13) to (4.19) are satisfied altogether by only one value from these distinct values i.e., = . Hence = .

So, *zp* = implies that = and *V* =

Also, = implies that =

And = implies that = .

Hence, strategies for player A (, ) and value of game = .

Hence by Fourier Motzkin Elimination Technique the optimal solution of game problem is: -

value of game = .

strategies for player A (, ) and strategies for player B (, ).

**2.** Solve the following game problem:

 Player B

 PlayerA

Now let us assume that *p1* & *p2*are probabilities for player A and *q1*, *q2* & *q3* are probabilities for player B to choose strategies for maximum gain of player A and minimum loss of player B.

**Formulation of LPP for player B:**

Let *V* be the maximum gain of player A. Then for minimum loss of player B we have to minimize *V*. So, LPP for player B can be written as

 Max. *zq =*   *+* +

Subject to + 3 + 11 ≤ 1

 8+ 5+ 2≤ 1

 and *,*≥ 0

Where, *=* , *=* and *=*

and min. *V* = max. = max. *zq.* So,

Now convert the above LPP in standard form for apply FMET, we get

 Max. *zq*

*zq - - - ≤* 0… (4.23)

 *+* 3+ 11 *≤* 1… (4.24)

8 *+* 5 + 2 *≤* 1… (4.25)

*- ≤* 0… (4.26)

 *-≤* 0 … (4.27)

 *- ≤* 0 … (4.28)

In the first stage of Fourier Motzkin elimination technique, we eliminate . For this we construct pairwise disjoint sets of inequalities for variable . i.e.,

 = {(4.24), (4.25)} = Two constraints

 = {(4.23), (4.26)} = Two constraints

 = {(4.27), (4.28)} = Two constraints

we will combine each inequality of with each inequality of . We get,

 Max*. zq*

 *zq* + 2 + 10 ≤ 1 … (4.29)

 3 + 11 ≤ 1 … (4.30)

 8*zq* - 3 - 6≤ 1 … (4.31)

 5+ 2≤ 1 … (4.32)

 *-* ≤ 0 … (4.33)

 *- ≤* 0 … (4.34)

Now, for elimination of , we construct pairwise disjoint sets of inequalities. So, let us construct , and . i.e.,

 = {(4.29), (4.30), (4.32)} = Three constraints

 = {(4.31), (4.33)} = Two constraints

 = {(4.34)} = One constraint

Now we combine inequalities of and for elimination of . We get,

 Max*. zq*

 19*zq* +18 5 … (4.35)

 *zq* + 10 1 … (4.36)

 8*zq* + 5 2 … (4.37)

 11 1 … (4.38)

 5*zq* - 3 1 … (4.39)

 11 1 … (4.40)

the remaining third variable will be eliminate with the help of pairwise disjoint sets of inequalities. So, let us construct , and .

 = {(4.35), (4.36), (4.37), (4.38), (4.40)} = Five constraints

 = {(4.39)} = Two constraints

 = { } = empty set

Now we combine inequalities of and for elimination of .

Combination of inequalities ofand gives different bounded values of *zq*. i.e.,

 *zq* ≤  … (4.41)

 *zq* ≤ … (4.42)

 *zq* ≤ … (4.43)

 *zq* ≤ … (4.44)

 *zq* ≤ … (4.45)

Here, we get four bounded values of *zq*. From these four values only *zq* = satisfies inequalities (4.35) to (4.40) altogether and gives = .

Now, putting *zq* = and = in inequalities (4.29) to (4.34), we find distinct bounded values for *y2*. Inequalities (4.29) to (4.34) are satisfied altogether by only one value from these distinct values i.e., = . So, *zq* = , = and = gives = 0 by back substitution from inequalities (4.23) to (4.28).

Hence, *zq* = implies that = and *V* = .

Also, = implies that  = 0, = implies that  =

And = implies that  = .

Hence, the strategies for player B (, ) and value of game = .

**Formulation of LPP for player A:**

LPP for player A is dual of LPP of player B. So, we have

 Min. *zp = +*

Subject to + 8 ≥ 1

 3+ 5 ≥ 1

 11 + 2 ≥ 1

 and *,*≥ 0

where = and  = .

Again, we make standard form for apply FMET on above LPP. We have

 Min. *zp*

*zp - -*  ≥ 0 … (4.46)

  *+* 8≥ 1 … (4.47)

 3+ 5≥ 1 … (4.48)

  *+* 2≥ 1 … (4.49)

 ≥ 0 … (4.50)

 ≥ 0… (4.51)

we will choose for elimination. For this, we will construct pairwise disjoint sets of inequalities for variable . So, pairwise disjoint sets are as follow:

 = {(4.47), (4.48), (4.49), (4.50)} = Four constraints

 = {(4.46)} = One constraint

 = {(4.51)} = One constraint

Now, we combine each inequality of with every inequality of . We get,

 Min. *zp*

 *zp* + 7 ≥ 1 … (4.52)

 3*zp* + 2 ≥ 1 … (4.53)

 11*zp* - 9≥ 1 … (4.54)

 *zp* - ≥ 0 … (4.55)

 ≥ 0 … (4.56)

we will eliminate the remaining second variable with the help of pairwise disjoint sets of inequalities. So, let us construct , and .

 = {(4.52), (4.53), (4.56)} = Three constraints

 = {(4.54), (4.55)} = Two constraints

 = { } = empty set

Now we combine inequalities of and for elimination of . We get,

 Min. *zq*

 *zp* ≥  … (4.57)

 *zp ≥* … (4.58)

 *zp* ≥ … (4.59)

 *zp ≥* … (4.60)

 *zp ≥* … (4.61)

 *zp ≥* 0 … (4.62)

Here, we get six bounded values of *zp*. From these six values only *zp* = satisfies inequalities (4.52) to (4.56) altogether and gives = .

Now, putting *zp* = and  = in inequalities (4.46) to (4.51), we find distinct bounded values for . Inequalities (4.46) to (4.29) are satisfied altogether by only one value from these distinct values i.e., = .

So, *zp* = implies that = and *V* =

Also, = implies that =

And = implies that = .

Hence, the strategies for player A (, ) and value of game = .

Hence by Fourier Motzkin Elimination Technique the optimal solution of cited game problem is:

Value of game = .

Strategies for player A (, ).

Strategies for player B (, ).

**5. Conclusion and Future Scope :**

The technique discussed in this chapter is a novel approach to solve the game problem. Different researchers used different methods to solve game problem. for example, traditional simplex method or graphical method. We applied Fourier Motzkin Elimination Technique on game problem. In this method, we are free to choose eliminating variable. Here, the proposed technique is based upon the solution of inequalities. When we combine the inequalities of first class with inequalities of second class , then the calculation becomes very easy. In this chapter, we applied Fourier Motzkin elimination technique on 2×2 and 2×n game problem and conclude that one can also be solved game problem by Fourier Motzkin Elimination Technique (FMET). Also, we compared our solution by graphical method and simplex method. This new approach to solve the game problem is simple and least time consuming. In future, we recommend to apply this elimination technique on various type of game problems

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