A Novel Study on the Generalized Stability in the Ulam-Hyers Sense for Quadratic Functional Equations Emerging from Geometrical Constructs

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ABSTRACT

In this work, the authors rigorously demonstrate the generalized Ulam–Hyers stability of a system of quadratic functional equations, intricately formulated from the Leibniz relation in Euclidean geometry and the median theorem of a triangle, within the sophisticated structure of intuitionistic fuzzy Banach spaces, employing Hyers’ foundational direct method.

Keywords— Quadratic functional systems, generalized Ulam–Hyers-type stability, intuitionistic fuzzy Banach frameworks, Hyers' direct analytical technique, geometric structures, classical Leibniz identity in Euclidean geometry, median configuration of triangular constructs.

#  INTRODUCTION

 The study of stability in functional equations traces its origin to a fundamental question posed by Ulam [36,37] regarding the stability of group homomorphisms. This query received an affirmative response in the context of Banach spaces through the pioneering work of Hyers [20]. Subsequently, this line of investigation was significantly generalized, yielding notable contributions by several researchers (see [3,18,26,29,32]).

 The general solutions and the generalized Ulam–Hyers–Rassias stability of quadratic functional equations have been explored by various authors, including Cholewa [15], S. Czerwik [16], Jung [23], and Ravi [30,31]. Over the past eight decades, these foundational problems have been extensively studied, with numerous forms of functional equations analyzed and solved through diverse mathematical approaches (see [1–2,4–14,17,21–22,24–25,27–28] and the references therein).

 Geometry, a classical and foundational branch of mathematics, is principally concerned with the properties of space, including aspects such as distance, shape, size, and the relative positioning of figures. Alongside arithmetic, it is among the oldest mathematical disciplines. A scholar specializing in this domain is known as a **geometer**. Until the advent of the 19th century, the study of geometry was predominantly confined to the Euclidean framework, wherein core concepts such as point, line, plane, distance, angle, surface, and curve formed the basis of exploration.

 Initially developed to represent the physical universe, geometry now finds applications across nearly all scientific domains, as well as in fields like art, architecture, and graphic design. Beyond its classical scope, geometry has profound influence in seemingly unrelated areas of mathematics. A striking example is the geometric foundation of general relativity, which is rooted in non-Euclidean geometry.

 Throughout the late 19th century and beyond, a series of transformative discoveries radically redefined the landscape of geometry, elevating it from a classical study of shapes and distances to a multifaceted and rigorously structured mathematical discipline. This profound evolution gave rise to numerous specialized branches, each rooted in distinct theoretical frameworks—most notably, **differential geometry**, **algebraic geometry**, **computational geometry**, **algebraic topology**, and **discrete geometry**.

 Concurrently, several non-Euclidean frameworks emerged, each systematically omitting specific Euclidean axioms to explore broader geometric phenomena. **Projective geometry**, for instance, focuses solely on the collinearity of points, deliberately ignoring distance and parallelism. **Affine geometry** abandons both angle and distance, while **finite geometry** dispenses with continuity altogether, offering powerful tools in combinatorics and logic.

 This intellectual expansion fundamentally altered the concept of space, which was once restricted to the three-dimensional physical continuum modeled by Euclidean principles. In the modern mathematical context, the term geometric space or simply space now denotes an abstract, axiomatically defined structure wherein geometric relations are rigorously encoded. For a comprehensive exposition of these developments, the reader is referred to [38].

The principal objective of this study is to translate fundamental geometrical properties into corresponding functional equations, and to rigorously investigate the conditions under which their solutions preserve and reflect these geometrical characteristics

**A. MEDIAN OF A TRIANGLE**

 In geometry, a median of a triangle is a line segment joining a vertex to the midpoint of the opposing side. Every triangle has exactly three medians: one running from each vertex to the opposite side. In the case of isosceles and equilateral triangles, a median bisects any angle at a vertex whose two adjacent sides are equal in length.



In a above triangle with the sides  and  the median drawn to the side  has the length of

 . (1.1)

**B. LEIBNIZ QUADRATIC FORMULA IN EUCLIDEAN GEOMETRY**

Let be an arbitrary point lying on the plane of the triangle  and  is the centroid (= Gravity center) of, then

 . (1.2)



The functional equation presented in (1.2) can be reformulated into a quadratic functional equation characterizing the median of a triangle, expressed as follows  (1.3)

Likewise, equation (1.4) can be equivalently reformulated as a quadratic functional equation representing the centroid of a triangle  is set by

  (1.4)

This study investigates the generalized Ulam–Hyers stability of the quadratic functional equations (1.3) and (1.4), which originate from the Leibniz formula in Euclidean geometry and the median of a triangle, respectively. The analysis is conducted within the setting of intuitionistic fuzzy Banach spaces through the application of Hyers’ direct method

**II BASIC DEFINITIONS RELATED TO INTUITIONISTIC FUZZY NORMED SPACES**

 In this section, we present fundamental definitions and notations pertaining to intuitionistic fuzzy normed spaces, as outlined in references [34], [35], and [36].

**Definition 2.1.** A binary operation \* :  is said to be a continuous t-norm if it satisfies the following conditions :

 \* is associative and commutative,

 \* is continuous,

  for all ,

  whenever and for each 

**Definition 2.2.** A binary operation  :  is said to be a continuous t-conorm if it satisfies the following conditions :

 is associative and commutative,

 is continuous,

  for all ,

  whenever and for each 

**Definition 2.3.** The five-tuple is said to be an intuitionistic fuzzy normed space (for short, IFNS) if is a vector space, \* is a continuous t-norm, is a continuous t-conorm,  are fuzzy sets on  such that for all and satisfying the following conditions :

[IFNS1] 

[IFNS2] 

[IFNS3]  if and only if 

[IFNS4]  for each 

[IFNS5] 

[IFNS6] is continuous,

[IFNS7] and

[IFNS8] 

[IFNS9]  if and only if 

[IFNS10]  for each 

[IFNS11] 

[IFNS12] is continuous,

[IFNS13] and.

In this case  is called an intuitionistic fuzzy norm

**Example 2.3.** Let  be a normed space,  and **** for all  For all , , , consider  and  then is an IFNS.

**Definition 2.5.** Let be an IFNS. Then, a sequence is said to be intuitionistic fuzzy convergent to if and  for all .

**Definition 2.6.** Let be an IFNS. Then, a sequence is said to be intuitionistic fuzzy Cauchy sequence if and for all and 

**Definition 2.7.** Let be an IFNS. Then,  is said to be Banach space if every intuitionistic fuzzy convergent in .

**III INTUITIONISTIC FUZZY STABILITY**

Henceforth, let  denote an intuitionistic fuzzy normed space, and let specifically represent an intuitionistic fuzzy Banach space. In this section, building upon a profound idea introduced by Gavruta, we rigorously establish the generalized stability of the functional equation within the framework pioneered by Hyers, Ulam, and Rassias. For brevity and clarity, we adopt the following notational convention for a given mappingand  by



and

for all .

**Theorem 3.1 :** Let  be fixed and let  are mappings for some  with  which adhere to the conditions

 (3.1)

   (3.2)

and  are functions conforming to the inequalities

  (3.3)

irrespective of the choice of  and all Consequently, a unique quadratic mappings  obeying (1.3) and (1.4) such that

 (3.4)

with respect to all and all  The mappings  are obtained by

 (3.5)

for all .

**Proof :** Assume . Taking the place of  by  in (3.3), using evenness of  and [IFNS4], it can be concluded that

 (3.6)

for every  and all . Altering  by  in (3.6), this leads to

  (3.7)

holds universally over  and all . Using [IFNS4], (3.1) in (3.7), it can be deduced

  (3.8)

for all  and all . Replacing  by  in (3.8), one can have

  (3.9)

irrespective of the choice of  and all . One can easy to see verify that

  (3.10)

for every. From equations (3.9) and (3.10), we have.

 (3.11)

for all and  where 

pertaining to all and all . Replacing  by  in (3.11) and using (3.1), [IFNS4], one can obtain

  (3.12)

for each and all  and all . Switching out  by  in (3.12), one can get

  (3.13)

across all and all  and all . Upon applying [IFNS4] to (3.13), we find

  (3.14)

pertaining to all and all  and all .

Since  and  According to the Cauchy Criterion for convergence in IFNS, we have that



are Cauchy sequences in  and it is complete, this sequence approaches a certain limit .This leads to the definition of the mapping  by



for each . Letting  in (3.14), deduce that

 (3.15)

holds for all and all  Letting  in (3.15) and using [IFNS6], this leads to

 (3.16)

pertaining to all and .

Now, our goal is to show that  accomplishes (1.3) and  meets (1.4), superseding  byin (3.3) corresponding fashion, one can obtain

 (3.18)

(3.19)

irrespective of the choice of and all  Now





Letting  in (3.18) and using (IFN7), (IFN13), (IFN3), (IFN9), we can see that  meets (1.3).

Proceeding similarly, one arrives at an additional assertion.

In order to establish the uniqueness of , let there be a functions  satisfies (1.3),  adheres to (1.4) and (3.4). Hence,

  (3.20)

with respect to all  and all . Owing to the fact that,



for each  and by [IFNS5] and [IFNS13], it is evident that



irrespective of any particular choice of and all . Therefore



holds universally over and all .Hence,



Therefore  are unique. Thus, the assertion holds true for .

Now,  by  in (3.6) , by invoking [IFNS4], it follows that  (3.21)

applicable to every and all . Since the argument mirrors that of the earlier case, the result is established for , and the theorem stands proved.

**Corollary 3.2.** Suppose that the functions  adhere to the prescribed inequalities  (3.22)

holds true for any  and all  Hence, one can assert the existence of a single quadratic functions  conforming to (1.7) for which

 

  (3.23)

irrespective of any particular choice of and all 

**VI CONCLUSIONS**

This study interprets certain geometric structures through the lens of quadratic-type functional relationships and rigorously examines their generalized Ulam-Hyers stability behavior within intuitionistic fuzzy Banach environments, adopting Hyers’ classical direct analytical strategy.

##### REFERENCES

1. J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York/London, 1966.
2. J. Aczel and J. Dhombres, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.
3. T. Aoki, “On the stability of the linear transformation in Banach spaces”, J. Math. Soc. Japan, 2 (1950), 64-66.
4. M. Arunkumar, Three Dimensional Quartic Functional Equation In Fuzzy Normed Spaces, Far East Journal of Applied Mathematics, 41(2), (2010), 83-94.
5. M. Arunkumar, G. Ganapathy, S. Murthy, S. Karthikeyan, Stability of the Generalized Arun-additive functional equation in Intuitionistic fuzzy normed spaces, International Journal Mathematical Sciences and Engineering Applications, Vol.4, No. V, (2010), 135-146.
6. M. Arunkumar, S. Jayanthi, S. Hema Latha Stability of quadratic derivations of Arun-quadratic functional equation, International Journal Mathematical Sciences and Engineering Applications, Vol. 5, No. V, (Sept. 2011), 433-443.
7. M. Arunkumar, S. Ramamoorthi, Solution and stability of a quadratic functional Equation in Banach space and its application, Far East Journal of Dynamical Systems, Vol 18, No 1, (2012), 21 – 32.
8. M. Arunkumar, S. Karthikeyan, Solution and Intuitionistic Fuzzy stability of n- dimensional quadratic functional equation: Direct and Fixed Point Methods, International Journal of Advanced Mathematical Sciences (IJAMS), Vol 2 (1), (2014), 21-33.
9. M. Arunkumar, T. Namachivayam , Intuitionistic Fuzzy Stability of a n- Dimensional Cubic Functional Equation: Direct and Fixed Point Methods, Intern. J. Fuzzy Mathematical Archive, Vol. 7, No. 1, (2015), 1-11 .
10. M. Arunkumar, John M Rassias, S. Karthikeyan, Stability Of A Leibniz Type Additive And Quadratic Functional Equation In Intuitionistic Fuzzy Normed Spaces, Advances in Theoretical and Applied Mathematics, Volume 11, Number 2 (2016), pp. 145-169.
11. M. Arunkumar, J.M. Rassias, G. Ganapathy, Leibniz Type AQ – Functional Equation, International Journal of Mathematics And its Applications, Volume 4, Issue 2 - C (2016), 149-169.
12. M. Arunkumar , S. Karthikeyan, Intuitionistic Random Stability of a Quadratic Functional Equation Originating from Sum of The Medians of a Triangle, International Journal of Difference Equations, Volume 12, Number 1 (2017), 55-70.
13. M. Arunkumar and E. Sathya, Hyers type Fuzzy stability of a radical reciprocal quadratic functional equation originating from 3 dimensional Pythagorean Means, International Journal Fuzzy Mathematical Archive, Vol. 14, No. 1, 2017, 23-33 .
14. M.Arunkumar, E. Sathya, T. Namachivayam, Ulam - Hyers Stability of Euler - Lagrange Quadratic Functional Equation in Intuitionistic Fuzzy Banach Spaces: Direct and Fixed Point Methods, ICIFSATA – 2018 International Journal of Current Advance Research Vol. 7, Issue 1 (2018), 99- 108.
15. P.W. Cholewa, Remarks on the stability of functional equations, Aequationes Math. 27 (1984)76–86.
16. S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ.Hamburg 62 (1992) 59–64.
17. G.L. Forti, Hyers–Ulam stability of functional equations in several variables, AequationesMath. 50 (1995) 143–190.
18. P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184 (1994), 431-436.
19. P. Gavruta, An answer to a question of John M. Rassias concerning the stability of Cauchyequation, in: Advances in Equations and Inequalities, in: Hadronic Math. Series, 1999, pp. 67–71.
20. D.H. Hyers, On the stability of the linear functional equation”, Proc.Nat. Acad.Sci.,U.S.A.,27 (1941) 222-224.
21. D.H. Hyers, The stability of homomorphisms and related topics, in: Global Analysis-Analysison Manifolds, Teubner—Texte zur Mathematik, Vol. 57, 1983, pp. 140–153.
22. D.H. Hyers, G. Isac,Th.M. Rassias, Stability of functional equations in several variables, Birkhauser, Basel, 1998.
23. S.M. Jung, On the Hyers–Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl. 222 (1998) 126–137.
24. Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.
25. V. Nasiri, M. Arunkumar, E.Sathya, T. Namachivayam, Functional Inequalities For A Generalized Quadratic Functional Equation In Various Banach Spaces, Malaya Journal of Mathematic, Vol. 9, No. 1, 2021, 966-980.
26. J.M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. USA, 46, (1982) 126-130.
27. John. M. Rassias, M. Arunkumar, S. Karthikeyan, Stability of a quadratic functional equation originating From sum of the medians of a triangle in fuzzy ternary Banach algebras: direct and fixed point methods, International Journal of Mathematics and its Applications , Volume 3, Issue 4 - A (2015), 45 -64.
28. Matina J. Rassias, M. Arunkumar, S. Ramamoorthi, Stability of the Leibniz additive-quadratic functional equation in Quasi-Beta normed space: Direct and fixed point methods, Journal Of Concrete And Applicable Mathematics (JCAAM), Vol. 14 No. 1-2, (2014), 22 – 46.
29. K.Ravi and M.Arunkumar, On The Generalized Hyers-Ulam-Rassias Stability of Quadratic Functional Equations, International Journal of Pure and Applied Mathematics Vol 28, No. 1, (2006) , 85-94.
30. K.Ravi, M.Arunkumar, R.Murali, On the Generalized Hyers –Ulam-Rassias Stability of a quadratic Functional Equations, Journal of Inequalites Pure and Applied Mathematics, Vol 9, Issue-1, (2008), 1-5.
31. K. Ravi, M. Arunkumar and J.M. Rassias, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.
32. Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc.Amer.Math. Soc., 72 (1978), 297-300.
33. R. Saadati, J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos, Solitons and Fractals. 27 (2006), 331–344.
34. R. Saadati, J. H. Park, Intuitionstic fuzzy Euclidean normed spaces, Commun. Math. Anal., 1 (2006), 85–90.
35. R. Saadati, S. Sedghi and N. Shobe, Modified intuitionistic fuzzy metric spaces and some fixed point theorems, Chaos,Solitons and Fractals, 38 (2008), 36–47.
36. S.M. Ulam, A Collection of Mathematical Problems, Interscience Publishers, New York, 1968,p. 63.
37. S.M. Ulam, Problems in Modern Mathematics, Science Editions,Wiley, NewYork, 1964 (Chapter VI, Some Questions in Analysis: 1, Stability).
38. https://en.wikipedia.org/wiki/Geometry